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Lie symmetries of a generalised non-linear Schrödinger equation: II. Exact solutions

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Abstract. We obtain group-invariant solutions of the non-linear equation $i\psi_t + \Delta\psi = a_0\psi + a_1\psi|\psi|^2 + a_2\psi|\psi|^4$ for which the symmetry group was previously shown to be the extended Galilei group for $a_1 a_2 \neq 0$ and the extended Galilei-similitude group for $a_1 = 0$ or $a_2 = 0$. We use the symmetry subgroups to reduce the equation to ordinary differential equations which are solved, whenever possible, with the help of a singularity analysis. Solutions are obtained in terms of elementary functions, Jacobi elliptic functions and Painlevé transcendents.

1. Introduction

This paper is the second in a series devoted to the determination of group-invariant solutions of a generalised non-linear Schrödinger equation (GNLSE) in 3 + 1 dimensions:

$$\begin{aligned} & i\psi_t + \Delta\psi = a_0\psi + a_1\psi|\psi|^2 + a_2\psi|\psi|^4 \\ & \psi \equiv \psi(x, y, z, t) \in \mathbb{C} \qquad a_i \in \mathbb{R} \qquad i = 1, 2, 3 \qquad (a_1, a_2) \neq (0,0) \end{aligned} \tag{1.1}$$

where Δ is the three-dimensional Laplace operator in Euclidean 3-space and a_i are constants.

In the first paper [1] we gave the group theoretical preliminaries which can be summarised as follows. The GNLSE (1.1) is, for $a_1 a_2 \neq 0$, invariant under the extended Galilei group $\tilde{G}(3) \equiv \tilde{G}$ defined by equation (2.3) in [1]. For $a_1 = 0$ or $a_2 = 0$, it is invariant under a larger group, namely the extended Galilei-similitude group $\tilde{G}^d(3) \equiv \tilde{G}^d$, including dilations (see [1], equation (2.4)). We also classified all subalgebras of the corresponding Lie algebras $\tilde{g}$ and $\tilde{g}^d$ into conjugacy classes and determined their isomorphism classes.

Our aim here is to apply the technique of symmetry reduction to this equation in order to obtain exact solutions. The method, going back to Lie [2] and described in various books [3-6], has been explained in [1]. We recall that the essential point is to rewrite (1.1) in terms of invariants of a subgroup of the symmetry group in order to reduce the number of independent variables in the equation. We shall mainly concentrate on subgroups with generic orbits of codimension one in the set of independent variables (x, y, z, t) and of codimension three in the union set of independent and dependent variables $(x, y, z, t, \psi, \psi^*)$. The reduced equations will in this case be

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ordinary differential equations (ODE) which can, in some cases, be integrated with the help of a Painlevé singularity analysis [7, 8]. In cases when they turn out to be of Painlevé type they can be solved in terms of elementary functions, Jacobi elliptic functions or Painlevé transcendents.

The paper is organised as follows. In § 2, we give the reduced equations for symmetry subgroups with orbits of codimension two without going into them in any further detail. In § 3, we concentrate on subgroups with orbits of codimension one and zero which reduce (1.1) to ordinary or algebraic equations. ODE of order one and algebraic equations are easily solved. ODE of order two are submitted to the Painlevé test and integrated explicitly, whenever possible. Translationally invariant solutions and solutions invariant under subgroups involving dilations (similarity solutions) are given in §§ 4 and 5, respectively. Cylindrically and spherically invariant solutions have also been obtained and can be found in separate papers [9, 10].

2. Symmetry reduction to partial differential equations in two variables

The general procedure used to perform the symmetry reduction using some specific subgroup G_0 of the symmetry group is to first find the invariants of G_0 and rewrite (1.1) in terms of them. These invariants are obtained by solving the equations

$$X_i Q(x, y, z, t, \psi, \psi^*) = 0 \qquad i = 1, \ldots, l \tag{2.1}$$

where $\{X_i\}$ is some basis for the Lie algebra of G_0 and Q is an auxiliary function.

In this section, we will restrict to subgroups with generic orbits of codimension two in the spacetime (x, y, z, t) and of codimension four in the space $(x, y, z, t, \psi, \psi^*)$. We then always have four invariants that can be chosen in the form

$$\begin{aligned} &I_1 = \xi_1(\boldsymbol{x}, t) \qquad I_2 = \xi_2(\boldsymbol{x}, t) \qquad I_3 \equiv f = \psi\alpha^{-1}(\boldsymbol{x}, t) \\ &I_4 \equiv f^* = \psi^* \alpha^{-1*}(\boldsymbol{x}, t). \end{aligned} \tag{2.2}$$

This permits us to reduce (1.1) to a partial differential equation for $f(\xi_1, \xi_2)$ which is a function of the two symmetry variables ξ_1 and ξ_2 and write the solution of the GNLSE (1.1) as

$$\psi(\boldsymbol{x}, t) = f(\xi_1, \xi_2)\alpha(\boldsymbol{x}, t) \tag{2.3}$$

where α, ξ_1 and ξ_2 are known.

Using the tables in [1] we obtain the following results for all the subgroups with orbits of codimension two.

(i) (j_1, j_2, j_3).

$$\begin{aligned} &\psi(\boldsymbol{x}, t) = f(r, t) \qquad r = (x^2 + y^2 + z^2)^{1/2} \\ &\mathrm{i} f_t + f_{rr} + (2/r) f_r = a_0 f + a_1 f |f|^2 + a_2 f |f|^4. \end{aligned} \tag{2.4}$$

(ii) $(j_3 + c p_3 + b m, t + (a - a_0) m)$, $a \in \mathbb{R}$, $b \geq 0$, $c \geq 0$.

$$\begin{aligned} &\psi(\boldsymbol{x}, t) = f(\rho, \xi) \exp(-\mathrm{i} a t) \exp(\mathrm{i} b \theta) \\ &\rho = (x^2 + y^2)^{1/2} \qquad \theta = \tan^{-1}(y/x) \qquad \xi = z + c\theta \\ &f_{\rho\rho} + (1/\rho) f_\rho + (1 + c^2/\rho^2) f_{\xi\xi} + (b/\rho^2)(2\mathrm{i} c f_\xi - b f) = (a_0 - a) f + a_1 f |f|^2 + a_2 f |f|^4. \end{aligned} \tag{2.5}$$

(iii) $(j_3+b(t-a_0m)+am, p_3)$, $a>0$, $b\in\mathbb{R}$ or $a=0$, $b\geqslant 0$.

$$\psi(\boldsymbol{x},t)=f(\rho,\xi)\exp(ai\theta) \qquad \xi=t+b\theta$$
$$\mathrm{i}f_\xi+(b^2/\rho^2)f_{\xi\xi}+(a/\rho^2)(2\mathrm{i}bf_\xi-af)+f_{\rho\rho}+(1/\rho)f_\rho=a_0f+a_1f|f|^2+a_2f|f|^4. \tag{2.6}$$

(iv) (k_2+ap_1, p_3), $a\geqslant 0$.

$$\psi(\boldsymbol{x},t)=f(t,\xi)\exp(\mathrm{i}y^2/4t) \qquad \xi=x-(a/t)y$$
$$\mathrm{i}f_t+(1+a^2/t^2)f_{\xi\xi}+(\mathrm{i}/2t)f=a_0f+a_1f|f|^2+a_2f|f|^4. \tag{2.7}$$

(v) $(t-a_0m+ak_3, p_1)$, $a>0$.

$$\psi(\boldsymbol{x},t)=f(y,\xi)\exp[\tfrac{1}{6}\mathrm{i}at(3z-at^2)] \qquad \xi=z-\tfrac{1}{2}at^2$$
$$f_{yy}+f_{\xi\xi}-\tfrac{1}{2}a\xi f=a_0f+a_1f|f|^2+a_2f|f|^4. \tag{2.8}$$

(vi) $(k_1+bp_2, k_2+bp_1+cp_2+ap_3)$, $a>0$, $b\geqslant 0$, $c\geqslant 0$ or $a=b=0$, $c\geqslant 0$.

$$\psi(\boldsymbol{x},t)=f(t,\xi)\exp\{(\mathrm{i}/4t)\,[x^2+(y^2t^2-2bxyt+b^2x^2)/(t^2-b^2+ct)]\}$$
$$\xi=a[(bx-yt)/(t^2-b^2+ct)]+z \tag{2.9}$$
$$\mathrm{i}f_t+\{1+[a^2(b^2+t^2)]/(t^2-b^2+ct)\}f_{\xi\xi}$$
$$+(\mathrm{i}/2t)[1+(b^2+t^2)/(t^2-b^2+ct)]f=a_0f+a_1f|f|^2+a_2f|f|^4.$$

(vii) $(j_3+bm,\ t-a_0m+ak_3)$, $a>0$, $b\in\mathbb{R}$.

$$\psi(\boldsymbol{x},t)=f(\rho,\xi)\exp\{\mathrm{i}[b\theta+\tfrac{1}{6}at(3z-at^2)]\} \qquad \xi=z-\tfrac{1}{2}at^2$$
$$f_{\rho\rho}+(1/\rho)f_\rho-(b^2/\rho^2)f+f_{\xi\xi}-\tfrac{1}{2}a\xi f=a_0f+a_1f|f|^2+a_2f|f|^4. \tag{2.10}$$

(viii) (j_3+bm, k_3), $b\geqslant 0$.

$$\psi(\boldsymbol{x},t)=f(t,\rho)\exp\{\mathrm{i}[b\theta+(z^2/4t)]\}$$
$$\mathrm{i}f_t+(1/\rho)f_\rho+f_{\rho\rho}-(b^2/\rho^2)f+(\mathrm{i}/2t)f=a_0f+a_1f|f|^2+a_2f|f|^4. \tag{2.11}$$

(ix) $(t+(a-a_0)m, p_3)$, $a\in\mathbb{R}$.

$$\psi(\boldsymbol{x},t)=f(x,y)\exp(-\mathrm{i}at)$$
$$f_{xx}+f_{yy}=(a_0-a)f+a_1f|f|^2+a_2f|f|^4. \tag{2.12}$$

(x) (p_1, p_2).

$$\psi(\boldsymbol{x},t)=f(t,z)$$
$$\mathrm{i}f_t+f_{zz}=a_0f+a_1f|f|^2+a_2f|f|^4. \tag{2.13}$$

(xi) $(j_3+am,\ d+bm)$, $a\geqslant 0$, $b\geqslant 0$.

$$\psi(x,t)=f(\xi_1,\xi_2)t^{-(\delta/2)}\exp[\mathrm{i}(a\theta-a_0t-\tfrac{1}{2}b\ln t)]$$

$$\xi_1=t/\rho^2 \qquad \xi_2=z/\rho \qquad \delta=\begin{cases}1 & \text{if } a_2=0,\ a_1\neq 0\\ \frac{1}{2} & \text{if } a_2\neq 0,\ a_1=0\end{cases}$$

$$\xi_1(\mathrm{i}+4\xi_1)f_{\xi_1}+\xi_1\xi_2f_{\xi_2}+4\xi_1^3f_{\xi_1\xi_1}+\xi_1(1+\xi_2^2)f_{\xi_2\xi_2}+4\xi_1^2\xi_2f_{\xi_1\xi_2}-(\tfrac{1}{2}b+a^2\xi_1+\tfrac{1}{2}\mathrm{i}\delta)f \tag{2.14}$$
$$=\begin{cases}a_1f|f|^2 & \text{with } \delta=1\\ a_2f|f|^4 & \text{with } \delta=\frac{1}{2}\end{cases}.$$

(xii) $(d+aj_3+bm,\ k_3)$, $a\geqslant 0$, $b\geqslant 0$.

$$\psi(x,t)=f(\xi_1,\xi_2)t^{-\delta/2}\exp\{\mathrm{i}[(z^2/4t)-\tfrac{1}{2}b\ln t-a_0t]\}$$

$$\xi_1=t/\rho^2 \qquad \xi_2=a\ln\rho+\theta$$

$$\xi_1(\mathrm{i}+4\xi_1)f_{\xi_1}+4\xi_1^3f_{\xi_1\xi_1}+\xi_1(1+a^2)f_{\xi_2\xi_2}-4a\xi_1^2f_{\xi_1\xi_2}+\tfrac{1}{2}[b+\mathrm{i}(1-\delta)]f \tag{2.15}$$
$$=\begin{cases}a_1f|f|^2 & \text{with } \delta=1\\ a_2f|f|^4 & \text{with } \delta=\frac{1}{2}\end{cases}.$$

(xiii) $(d+aj_3+bm,\ t)$, $a\geqslant 0$, $b\geqslant 0$.

$$\psi(x,t)=f(\xi_1,\xi_2)z^{-\delta}\exp[-\mathrm{i}(a_0t+b\ln z)] \qquad \xi_1=z/\rho \qquad \xi_2=a\ln\rho+\theta$$

$$\xi_1[\xi_1^2-2(\delta+b\mathrm{i})]f_{\xi_1}+\xi_1^2(1+a^2)f_{\xi_2\xi_2}-2a\xi_1^3f_{\xi_1\xi_2}+\xi_1^2(1+\xi_1^2)f_{\xi_1\xi_1}+(\delta+b\mathrm{i})(\delta+b\mathrm{i}+1)f \tag{2.16}$$
$$=\begin{cases}a_1f|f|^2 & \text{with } \delta=1\\ a_2f|f|^4 & \text{with } \delta=\frac{1}{2}\end{cases}.$$

(xiv) $(d+aj_3+bm,\ p_3)$, $a\geqslant 0$, $b\geqslant 0$.

$$\psi(x,t)=f(\xi_1,\xi_2)t^{-\delta/2}\exp[\mathrm{i}(-\tfrac{1}{2}b\ln t-a_0t)] \qquad \xi_1=t/\rho^2 \qquad \xi_2=a\ln\rho+\theta$$

$$\xi_1(\mathrm{i}+4\xi_1)f_{\xi_1}+4\xi_1^3f_{\xi_1\xi_1}+\xi_1(1+a^2)f_{\xi_2\xi_2}-4a\xi_1^2f_{\xi_1\xi_2}+\tfrac{1}{2}(b-\delta\mathrm{i})f \tag{2.17}$$
$$=\begin{cases}a_1f|f|^2 & \text{with } \delta=1\\ a_2f|f|^4 & \text{with } \delta=\frac{1}{2}\end{cases}.$$

This completes the list of all reductions to partial differential equations in two variables. We will not go into them in any further detail here, but just mention that (2.4) has been treated elsewhere [10] and that (2.13) for $a_2=0$ is the well known integrable non-linear (cubic) Schrödinger equation [11, 12].

3. Symmetry reduction to ordinary differential equations and algebraic equations

3.1. General comments

Subgroups of the symmetry group that have generic orbits of codimension one in the space of independent variables (x, y, z, t) and of codimension three in $(x, y, z, t, \psi, \psi^*)$ space will provide reductions of (1.1) to ODE.

The requirement that the solution $\psi(x, t)$ be invariant under such a subgroup will imply that the solution has the form

$$\psi(x,t)=f(\xi)\alpha(x,t) \qquad f(\xi)=M(\xi)\exp(i\chi(\xi)) \qquad M(\xi),\chi(\xi)\in\mathbb{R} \tag{3.1}$$

where $\alpha(\boldsymbol{x}, t)$ and $\xi(\boldsymbol{x}, t)$ are explicitly known for each subgroup. The complex function $f(\xi)$ satisfies a ODE, obtained by substituting (3.1) into (1.1).

The Lie algebra of the symmetry group is realised [1] by vector fields in the tangent space $(\partial_x, \partial_y, \partial_z, \partial_t, \partial_\psi, \partial_{\psi^*})$. Consider the Lie subalgebra corresponding to the subgroup providing the reduction. If the projections of the elements of this subalgebra onto the space tangent to the space of independent variables span the $(\partial_x, \partial_y, \partial_z)$ space, then the symmetry variable ξ in (3.1) will be

$$\xi = t \tag{3.2}$$

and the GNLSE (1.1) will be reduced to a first-order ODE. Separating out the real and imaginary parts of the corresponding ODE, we can always solve it explicitly.

In all other cases, subgroups of the considered type will lead to a second-order complex ODE for $f(\xi)$. Taking the real and imaginary parts of the equation separately, we obtain a system of two coupled second-order real equations for the modulus $M(\xi)$ and the phase $\chi(\xi)$ of $f(\xi)$.

For many subalgebras, we can directly solve one of the equations and obtain the phase in the form

$$\chi(\xi) = S_0 \int \frac{1}{M^2(\xi)} h(\xi)\, \mathrm{d}\xi + k(\xi) + \chi_0 \tag{3.3}$$

where χ_0 and S_0 are real constants and the functions $h(\xi)$ and $k(\xi)$ depend on the subalgebra under consideration. Using (3.3), we then obtain a second-order real ODE for $M(\xi)$ alone.

For the remaining subalgebras, we can also decouple the two real equations for $M(\xi)$ and $\chi(\xi)$. However, this introduces third-order non-linear ODE. In the present paper, we concentrate on the case of second-order decoupled ODE, leaving the third-order ones for the third paper of this series.

The GNLSE (1.1) is reduced to an algebraic equation, if we require that the solution should be invariant under a larger subgroup, having four-dimensional orbits in (x, y, z, t) space. There are of course many such subgroups. We shall restrict ourselves to those that provide non-trivial solutions that are not special cases of solutions obtained from first-order ODE, i.e. to subalgebras which do not contain a three-dimensional Abelian subalgebra that is a subspace of $(p_1, p_2, p_3, k_1, k_2, k_3)$.

3.2. Algebraic equations

We shall now run through the list of algebras providing algebraic equations yielding non-trivial solutions.

(i) (d, t, p_1, p_2).

$$\psi(\boldsymbol{x}, t) = (2/a_1)^{1/2}(1/z)\exp[\mathrm{i}(-a_0 t + \chi_0)] \qquad a_1 > 0,\ a_2 = 0 \tag{3.4a}$$

$$\psi(\boldsymbol{x}, t) = (3/4a_2)^{1/4}(1/\sqrt{z})\exp[\mathrm{i}(-a_0 t + \chi_0)] \qquad a_2 > 0,\ a_1 = 0. \tag{3.4b}$$

(ii) $(d, t, p_3, j_3 + am)$, $a \geqslant 0$.

$$\psi(\boldsymbol{x}, t) = \left(\frac{1-a^2}{a_1}\right)^{1/2} \frac{1}{\rho} \exp[\mathrm{i}(-a_0 t + a\theta + \chi_0)] \qquad a_2 = 0 \tag{3.5a}$$

$$\psi(\boldsymbol{x}, t) = \left(\frac{1-4a^2}{4a_2}\right)^{1/4} \frac{1}{\sqrt{\rho}} \exp[\mathrm{i}(-a_0 t + a\theta + \chi_0)] \qquad a_1 = 0 \tag{3.5b}$$

with

$$\rho=(x^2+y^2)^{1/2}.$$

The constant a must be chosen so that the expressions under the square root signs are positive.

(iii) $(d+am, j_1, j_2, j_3, t)$, $a\geqslant 0$.

$$\psi(x,t)=\left(\frac{-(1+4a^2)}{a_2}\right)^{1/4}\frac{1}{\sqrt{r}}\exp[\mathrm{i}(-a_0t-a\ln r+\chi_0)] \qquad a_1=0,\ a_2<0 \tag{3.6}$$

with

$$r=(x^2+y^2+z^2)^{1/2}.$$

No non-zero solutions exist for the cubic case ($a_2=0$).

3.3. First-order ODE

We run through all subalgebras, the elements of which are represented by vector fields spanning the $(\partial_x, \partial_y, \partial_z)$ space. We order them by the maximal number of space translations present in a basis. We only give the final result. Throughout, M_0 and χ_0 are arbitrary real constants.

(i) (p_1, p_2, p_3).

$$\psi(x,t)=M_0\exp[-\mathrm{i}(a_0+a_1M_0^2+a_2M_0^4)t+\mathrm{i}\chi_0]. \tag{3.7}$$

(ii) (k_3, p_1, p_2).

$$\psi(x,t)=M_0t^{-1/2}\exp\{\mathrm{i}[(z^2/4t)-a_0t-a_1M_0^2\ln t+a_2(M_0^4/t)+\chi_0]\}. \tag{3.8}$$

(iii) (k_1, k_2+ap_2, p_3), $a\geqslant 0$.

$$\psi(x,t)=M_0[t(t+a)]^{-1/2}\exp\left[\mathrm{i}\left(\frac{x^2}{4t}+\frac{y^2}{4(t+a)}-a_0t-a_1M_0^2\int\frac{\mathrm{d}t}{t(t+a)}\right.\right.$$
$$\left.\left.-a_2M_0^4\int\frac{\mathrm{d}t}{t^2(t+a)^2}+\chi_0\right)\right]. \tag{3.9}$$

The integrals in the phase are easy to calculate, but we refrain from evaluating them, in order to avoid distinguishing between the generic case and the special case $a=0$.

(iv) $(k_1+ap_1, k_2+bp_2, k_3)$, $a, b\in\mathbb{R}$.

$$\psi(x,t)=M_0[t(t+a)(t+b)]^{-1/2}\exp\left\{\mathrm{i}\left[\frac{1}{4}\left(\frac{x^2}{t+a}+\frac{y^2}{t+b}+\frac{z^2}{t}\right)-a_0t\right.\right.$$
$$\left.\left.-a_1M_0^2\int\frac{\mathrm{d}t}{t(t+a)(t+b)}-a_2M_0^4\int\frac{\mathrm{d}t}{t^2(t+a)^2(t+b)^2}\right]\right\}. \tag{3.10}$$

We again refrain from evaluating the integrals in the phase, since their evaluation would require the consideration of numerous special cases ($a=b$, $a=0$, etc).

3.4. Second-order ODE

In this subsection, we shall derive the second-order equations for the magnitude and phase of $\psi(x,t)$ and decouple them, whenever possible. The actual solution of the

ODE is left to §§ 4 and 5, and to a future paper. Constant solutions of the ODE will be pointed out whenever they lead to non-trivial expressions for $\psi(\boldsymbol{x}, t)$.

We list the subalgebras and expressions or equations for $\alpha(\boldsymbol{x}, t)$, $M(\xi)$ and $\chi(\xi)$ of (3.1).

(i) $(j_3+am, p_3, t+(b-a_0)m)$, $a\geqslant 0$, $b\in\mathbb{R}$.

$$\alpha = \mathrm{e}^{ai\theta}\,\mathrm{e}^{-ibt} \tag{3.11a}$$

$$\chi = S_0\int\frac{\mathrm{d}\rho}{\rho M^2}+\chi_0 \tag{3.11b}$$

$$\begin{aligned}&\ddot{M}-(S_0^2/\rho^2M^3)+(\dot{M}/\rho)-(a^2/\rho^2)M\\&\qquad=(a_0-b)M+a_1M^3+a_2M^5 \qquad \xi=\rho=(x^2+y^2)^{1/2}\end{aligned} \tag{3.11c}$$

where S_0 is a real constant. A constant solution $M(\xi)=M_0$ exists for $a=S_0=0$ and leads to

$$\psi(\boldsymbol{x}, t)=\{(1/2a_2)(-a_1\pm[a_1^2-4a_2(a_0-b)]^{1/2}\}^{1/2}\exp[\mathrm{i}(a\theta-bt)] \tag{3.11d}$$

which is valid as long as the expressions under the square roots are non-negative.

(ii) $(t-a_0m+ak_3, p_1, p_2)$, $a>0$.

$$\alpha=\exp[\tfrac{1}{6}\mathrm{i}at(3z-at^2)] \tag{3.12a}$$

$$\chi=S_0\int\frac{\mathrm{d}\xi}{M^2}+\chi_0 \tag{3.12b}$$

$$\ddot{M}-(S_0^2/M^3)-(a/2)\xi M=a_0M+a_1M^3+a_2M^5 \tag{3.12c}$$

$$\xi=z-\tfrac{1}{2}at^2. \tag{3.12d}$$

No constant solutions exist since $a\neq 0$.

(iii) $(t+(a-a_0)m, p_1, p_2)$, $a\in\mathbb{R}$.

$$\alpha=\mathrm{e}^{-\mathrm{i}at} \tag{3.13a}$$

$$\chi=S_0\int\frac{\mathrm{d}z}{M^2}+\chi_0 \tag{3.13b}$$

$$\ddot{M}-(S_0^2/M^3)=(a_0-a)M+a_1M^3+a_2M^5 \qquad \xi=z. \tag{3.13c}$$

Constant solutions $M(\xi)=M_0$ are given by the positive roots of

$$S_0^2+(a_0-a)M_0^4+a_1M_0^6+a_2M_0^8=0. \tag{3.14a}$$

Thus

$$\psi(\boldsymbol{x}, t)=M_0\exp\{\mathrm{i}[(S_0/M_0^2)z+\chi_0-at]\} \tag{3.14b}$$

where M_0 is given by (3.14a).

(iv) $(j_1, j_2, j_3, t+(b-a_0)m)$, $b\in\mathbb{R}$.

$$\alpha=\mathrm{e}^{-ibt} \tag{3.15a}$$

$$\chi=S_0\int\frac{\mathrm{d}r}{r^2M^2}+\chi_0 \tag{3.15b}$$

$$\begin{aligned}&\ddot{M}-(S_0^2/r^4M^3)+(2/r)\dot{M}+bM=a_0M+a_1M^3+a_2M^5\\&\xi=r=(x^2+y^2+z^2)^{1/2}.\end{aligned} \tag{3.15c}$$

A constant solution $M(\xi)=M_0$ exists for $S_0=0$ and leads to relation (3.11d) with $a=0$.

(v) $(d+bm, k_3, j_3+am)$, $a\geqslant 0$, $b\geqslant 0$.

$$\alpha = t^{-\delta/2}\exp\{\mathrm{i}[-a_0 t+(z^2/4t)+a\theta-\tfrac{1}{2}b\ln t]\} \tag{3.16a}$$

$$\xi = t/\rho^2. \tag{3.16b}$$

This subalgebra leads to third-order ODE. A constant solution $M(\xi)=M_0$ exists for $S_0=a=0$, $\delta=1$, namely

$$\psi(x,t)=(b/2a_1)^{1/2}t^{-1/2}\exp\{\mathrm{i}[-a_0 t-\tfrac{1}{2}b\ln t+(z^2/4t)+\chi_0]\} \tag{3.17}$$

where $b/a_1>0$.

(vi) $(d+bm, t, j_3+am)$, $a\geqslant 0$, $b\geqslant 0$.

$$\alpha = z^{-\delta}\exp[-\mathrm{i}(a_0 t-a\theta+b\ln z)] \tag{3.18a}$$

$$\xi = z/\rho. \tag{3.18b}$$

For the quintic case $(a_1=0)$ we obtain

$$\chi = S_0\int\frac{\xi\,\mathrm{d}\xi}{M^2(1+\xi^2)}+b\ln\left(\frac{\xi}{(1+\xi^2)^{1/2}}\right)+\chi_0 \tag{3.18c}$$

$$\xi^2(1+\xi^2)\ddot{M}+\xi(\xi^2-1)\dot{M}-\frac{S_0^2\xi^4}{1+\xi^2}\frac{1}{M^3}+\frac{b^2}{1+\xi^2}M+(\tfrac{3}{4}-b^2-a^2\xi^2)M=a_2M^5. \tag{3.18d}$$

For the cubic case $(a_2=0)$, we get a third-order ODE for $b>0$. However, for $b=0$ the result is simpler, namely

$$\chi = S_0\int\frac{\xi^2}{M^2(1+\xi^2)^{3/2}}\,\mathrm{d}\xi+\chi_0 \tag{3.18e}$$

$$\xi^2(1+\xi^2)\ddot{M}-\frac{\xi^6}{(1+\xi^2)^2}\frac{S_0^2}{M^3}+\xi(\xi^2-2)\dot{M}+(2-a^2\xi^2)M=a_1M^3. \tag{3.18f}$$

A constant solution $M(\xi)=M_0$ exists for $\delta=\frac{1}{2}$ $(a_1=0)$, $a=b=S_0=0$ and leads to (3.4b).

(vii) $(d+bm, p_3, j_3+am)$, $a\geqslant 0$, $b\geqslant 0$.

$$\alpha = t^{-\delta/2}\exp[\mathrm{i}(-a_0 t+a\theta-\tfrac{1}{2}b\ln t)] \tag{3.19a}$$

$$\xi = t/\rho^2. \tag{3.19b}$$

For the quintic case $(a_1=0)$, we obtain a third-order ODE. However, for the cubic case $(a_2=0)$, we get

$$\chi = S_0\int\frac{\mathrm{d}\xi}{\xi M^2}+\frac{1}{8\xi}+\chi_0 \tag{3.20a}$$

$$4\xi^3\ddot{M}+4\xi^2\dot{M}-4\xi(S_0^2/M^3)+[\tfrac{1}{2}b-a^2\xi+(1/16\xi)]M=a_1M^3. \tag{3.20b}$$

No constant solutions of (3.20b) exist.

(viii) $(d+bm, k_1, k_2)$, $b\geqslant 0$.

$$\alpha = t^{-\delta/2}\exp[\mathrm{i}(\rho^2/4t-a_0 t-\tfrac{1}{2}b\ln t)] \tag{3.21a}$$

$$\xi = t/z^2. \tag{3.21b}$$

This subalgebra leads to third-order ODE with no constant solution for $M(\xi)$.

(ix) $(d+bm, p_1, p_2)$, $b \geqslant 0$.

$$\alpha = t^{-\delta/2} \exp[-\mathrm{i}(a_0 t + \tfrac{1}{2} b \ln t)] \tag{3.22a}$$

$$\xi = t/z^2. \tag{3.22b}$$

For the quintic case $(a_1 = 0)$

$$\chi = S_0 \int \frac{\mathrm{d}\xi}{\xi^{3/2} M^2} + (1/8\xi) + \chi_0 \tag{3.22c}$$

$$4\xi^3 \ddot{M} - (4S_0^2/M^3) + \tfrac{1}{16}(M/\xi) + 6\xi^2 \dot{M} + \tfrac{1}{2} bM = a_2 M^5. \tag{3.22d}$$

For the cubic case $(a_2 = 0)$, we obtain a third-order ODE. No constant solution exists for $M(\xi)$.

(x) $(d + aj_3 + bm, t, p_3)$, $a \geqslant 0$, $b \geqslant 0$.

$$\alpha = \rho^{-\delta} \exp[-\mathrm{i}(a_0 t + b \ln \rho)] \tag{3.23a}$$

$$\xi = a \ln \rho + \theta. \tag{3.23b}$$

The case $b > 0$ yields third-order ODE. However, for $b = 0$ we obtain

$$\chi = S_0 \int M^{-2} \exp[2\delta a \xi/(a^2+1)]\, \mathrm{d}\xi + \chi_0 \tag{3.23c}$$

$$(a^2+1)\ddot{M} - (a^2+1) M^{-3} \exp[4\delta a\xi/(a^2+1)] S_0^2 - 2\delta a \dot{M} + \delta^2 M$$
$$= \begin{cases} a_2 M^5 & a_1 = 0,\ \delta = \frac{1}{2} \\ a_1 M^3 & a_2 = 0,\ \delta = 1. \end{cases} \tag{3.23d}$$

Constant solutions exist for $a = 0$. For the quintic case, we obtain (with $a_1 = 0$)

$$\psi(x, t) = \{(1/8a_2)[1 \pm (1 - 64 a_2 S_0^2)^{1/2}]\}^{1/4} \rho^{-1/2}$$
$$\times \exp \mathrm{i}[\![S_0 \theta \{(1/8a_2)[1 \pm (1 - 64 a_2 S_0^2)^{1/2}]\}^{-1/2} - a_0 t + \chi_0]\!] \tag{3.23e}$$

where S_0 is chosen such that $|\psi| > 0$. For the cubic case, we have

$$\psi(x, t) = M_0(1/\rho) \exp\{\mathrm{i}[(S_0/M_0^2)\theta - a_0 t + \chi_0]\} \tag{3.23f}$$

where M_0 is a positive root of

$$a_1 M_0^6 - M_0^4 + S_0^2 = 0. \tag{3.23g}$$

(xi) $(d + bm, k_3, p_1)$, $b \geqslant 0$.

$$\alpha = t^{-\delta/2} \exp\{\mathrm{i}[(z^2/4t) - \tfrac{1}{2} b \ln t - a_0 t]\} \tag{3.24a}$$

$$\xi = t/y^2. \tag{3.24b}$$

This subalgebra gives third-order ODE. A constant solution exists for the cubic case $(\delta = 1)$ and yields a special case of (3.8).

(xii) $(j_1, j_2, j_3, d + bm)$, $b \geqslant 0$.

$$\alpha = t^{-\delta/2} \exp[-\mathrm{i}(a_0 t + \tfrac{1}{2} b \ln t)] \tag{3.25a}$$

$$\xi = t/r^2. \tag{3.25b}$$

This subalgebra leads to third-order ODE and no constant solution exists for $M(\xi)$.

Our next task is to solve the second-order ODE for $M(\xi)$ obtained above. In general, this is a formidable task, since the equations are non-linear and quite complicated.

We use two systematic approaches. One consists of finding the symmetry group of the ODE, if one exists, and using it to decrease the order of the equation. The second method, which we found to be more fruitful, is to determine whether the ODE happens to belong to a class of integrable non-linear ODE, namely the class of Painlevé type equations [7, 13]. By definition, solutions of Painlevé type equations have no moving critical points (branch points or essential singularities that are functions of the integration constants).

Equations having the Painlevé property are in general much easier to integrate. In fact, Painlevé and Gambier have classified all equations of the form

$$\ddot{y} = F(x, y, \dot{y}) \tag{3.26}$$

where F is rational in y and $\dot{y}$ and analytic in x [7, 13], having no moving critical points. Furthermore, they have obtained the first integrals of 44 of the 50 equations in the representative list. The remaining 6 define the so-called Painlevé transcendents and they cannot be integrated in terms of known functions. However the other 44 can be solved in terms of elementary or Jacobi elliptic functions or reduced to the Riccati equation.

We first submit the equation to the 'Painlevé test' [8], in order to determine whether it satisfies certain necessary conditions for having the Painlevé property. This test has been implemented as a MACSYMA program [14]. If the equation passes the test then we look for a transformation of the form

$$y(x) = \frac{\alpha w(\zeta) + \beta}{\gamma w(\zeta) + \delta} \qquad \zeta = \zeta(x) \tag{3.27}$$

where α, β, γ and δ are functions of x, which takes the equation into one of the 50 standard forms.

Among the above second-order equations, those that do not pass the test for any value of the parameters involved are (3.18d), (3.18f), (3.20b) and (3.22d). Furthermore, no symmetries exists for these equations that could be used to reduce their order.

For all the other second-order equations, the test indicates that if we put

$$M = [H(\xi)]^{1/2} \tag{3.28}$$

then the equations for $H(\xi)$ can pass the test under certain conditions on the parameters. We will now concentrate on subalgebras (3.12) and (3.13) which provide translationally invariant solutions and subalgebra (3.23) which gives similarity solutions. Subalgebras (3.11) and (3.15), compatible with Cauchy conditions imposed on a cylinder and on a sphere, respectively, have been analysed in two separate papers [9, 10].

4. Translationally invariant solutions

4.1. General comments

In this subsection we shall consider solutions of the GNLSE (1.1), invariant under subgroups involving the two translations generated by p_1 and p_2. For all practical purposes, we are solving the GNLSE (2.13) in one space dimension. In order to obtain a ODE, we must add one more symmetry generator, acting non-trivially on the space (t, z, ψ, ψ^*). We are thus interested in three-dimensional subalgebras of the symmetry algebra, containing (p_1, p_2) as an ideal.

Equivalently, we may view this subsection as a group theoretical analysis of equation (2.13) as such. For $a_1 a_2 \neq 0$, this equation is invariant under the extended Galilei group $\tilde{G}(1)$, generated by (t, p_3, k_3, m). For $a_1 = 0$, $a_2 \neq 0$ or $a_1 \neq 0$, $a_2 = 0$, the equation is invariant under the extended Galilei-similitude group $\tilde{G}^d(1)$, containing dilations as well.

The generator that can be added to (p_1, p_2) in order to obtain a ODE is represented by one of the following: $d + am$, $t - a_0 m + ak_3 (a > 0)$, $t + (a - a_0)m$, k_3 or p_3. Using p_3 or k_3, we obtain one of the first-order ODE solved in § 3, i.e. (3.7) or (3.8), respectively. Using $d + am$ for $a_1 = 0$, $a_2 \neq 0$, we obtain (3.22*d*) which does not pass the Painlevé test. For $a_1 \neq 0$, $a_2 = 0$, we obtain a third-order equation (that does have the Painlevé property). Note that (2.13) for $a_2 = 0$ is integrable and that hence all ODE obtained from it have the Painlevé property.

Here we concentrate on the remaining two subalgebras, i.e.

$$(t - a_0 m + ak_3, p_1, p_2) \qquad a > 0 \tag{4.1}$$

$$(t + (a - a_0)m, p_1, p_2) \qquad a \in \mathbb{R}. \tag{4.2}$$

Both of these subalgebras lead to travelling-wave solutions, i.e. solutions for which the absolute value $M(\xi)$ of ψ is a function of a variable of the form

$$\xi = z + \eta(t)$$

where η is some (specific) function. Such solutions are usually the most important ones in physical applications. Using (4.1) and (4.2) to perform a symmetry reduction, we obtain equations (3.12) and (3.13), respectively.

As suggested by the Painlevé test, we make the transformation (3.28) and obtain respectively the following equations for $H(\xi)$:

$$\ddot{H} = (1/2H)\dot{H}^2 + 2S_0^2/H + a\xi H + 2a_0 H + 2a_1 H^2 + 2a_2 H^3 \tag{4.3}$$

$$\ddot{H} = (1/2H)\dot{H}^2 + 2S_0^2/H + 2(a_0 - a)H + 2a_1 H^2 + 2a_2 H^3. \tag{4.4}$$

Equation (4.3) passes the test for $a_2 = 0$ only. Equation (4.4) passes the test for all values of the parameters.

Under these conditions, they can be transformed into standard Painlevé type equations according to the general theory [7] by setting

$$H(\xi) = \lambda(\xi) W(\eta) > 0 \qquad \eta = \eta(\xi) \tag{4.5}$$

with $\lambda(\xi)$ and $\eta(\xi)$ appropriately chosen. The equation for $W(\eta)$ is then the standard one. Let us consider the cubic and quintic cases separately.

4.2. Solutions of the cubic NLSE *($a_2 = 0$) obtained using the algebra (4.1)*

To transform (4.3) (with $a_2 = 0$) into a standard form, we choose

$$\lambda(\xi) = \lambda_0 \neq 0 \tag{4.6a}$$

$$\eta = \varepsilon (a_1 \lambda_0 / 2)^{1/2} \xi + \eta_0 \tag{4.6b}$$

where λ_0 and η_0 are constants, and obtain

$$\ddot{W} = (1/2W)\dot{W}^2 + 4W^2 + \frac{2}{a_1 \lambda_0}\left[2a_0 + \varepsilon\left(\frac{2}{a_1 \lambda_0}\right)^{1/2} a(\eta - \eta_0)\right] W + \frac{4S_0^2}{a_1 \lambda_0^3}\frac{1}{W}. \tag{4.7}$$

First of all, equation (4.7) does not allow constant solutions for W since $a \neq 0$. Secondly, it represents three different cases.

(i) For $a_0 = S_0 = 0$, we set

$$\eta_0 = 0 \tag{4.8a}$$

$$a = (1/\sqrt{2})\varepsilon(\lambda_0 a_1)^{3/2} > 0 \tag{4.8b}$$

and obtain one of the 50 standard Painlevé type equations, namely the equation denoted PXX by Ince [7]:

$$\ddot{W} = (1/2W)\dot{W}^2 + 4W^2 + 2\eta W. \tag{4.9}$$

Putting

$$W = u^2 \tag{4.10}$$

we reduce it to the second Painlevé transcendant P_{II}

$$\ddot{u} = 2u^3 + \eta u. \tag{4.11}$$

The solution of the GNLSE (1.1) is then

$$\psi(x, t) = \lambda_0^{1/2} P_{\mathrm{II}}[\varepsilon(\tfrac{1}{2}a_1\lambda_0)^{1/2}\xi] \exp[\mathrm{i}\tfrac{1}{6}at(3z - at^2)] \exp(\mathrm{i}\chi_0) \qquad a_0 = 0 \tag{4.12}$$

where a and ξ are given by (4.8b) and (3.12d) respectively. For $a_1 > 0(\lambda_0 > 0)$, P_{II} must be chosen real; for $a_1 < 0(\lambda_0 < 0)$, P_{II} must be chosen to be pure imaginary.

(ii) For $a_0 \neq 0$ and $S_0 = 0$, we set

$$\lambda_0 = (4a)^{2/3}/a_1 \tag{4.13a}$$

$$\eta_0 = a_0(2/a)^{2/3} \tag{4.13b}$$

and again obtain (4.9). The solution of (1.1) then becomes

$$\psi(x, t) = (4a)^{1/3} a_1^{-1/2} P_{\mathrm{II}}[(a/2)^{1/3}\xi + a_0(4/a)^{1/3}] \exp[\mathrm{i}\tfrac{1}{6}at(3z - at^2)] \times \exp(\mathrm{i}\chi_0) \qquad a_0 \neq 0. \tag{4.14}$$

For $a_1 > 0$, P_{II} must be real; for $a_1 < 0$, P_{II} must be pure imaginary.

(iii) For the case $S_0 \neq 0$, we transform (4.3) (with $a_2 = 0$) into the standard [7] Painlevé equation PXXXIV by the choice

$$\lambda_0 = -2\mathrm{i}\varepsilon S_0/a^{1/3} \tag{4.15a}$$

$$\eta = \varepsilon(a_1\lambda_0/2\alpha)^{1/2}\xi - (2a_0/a^{2/3}) \tag{4.15b}$$

$$\alpha = -\mathrm{i}\varepsilon(a_1/a)S_0. \tag{4.15c}$$

We then obtain

$$\ddot{W} = (1/2W)\dot{W}^2 + 4\alpha W^2 - \eta W - 1/2W. \tag{4.16}$$

The general solution of (4.16) is also expressible in terms of the second Painlevé transcendent P_{II}. Indeed, putting

$$2\alpha W = \dot{V} + V^2 + \tfrac{1}{2}\eta \tag{4.17}$$

we find that V satisfies

$$\ddot{V} = 2V^3 + \eta V - 2\alpha - \tfrac{1}{2} \tag{4.18}$$

which is solved by $V = P_{\mathrm{II}}(\eta)$.

This completes the discussion for the subalgebra (4.1).

4.3. Solutions of cubic NLSE *obtained using the algebra (4.2)*

To transform (4.4) (with $a_2=0$) into standard form, we use the relations (4.5) and (4.6a, b) (with $\xi\equiv z$) and obtain, for W,

$$\ddot{W}=(1/2W)\dot{W}^2+4W^2+(4/a_1\lambda_0)(a_0-a)W+(4S_0^2/a_1\lambda_0^3)(1/W). \quad (4.19)$$

Relation (4.19) admits constant solutions $W(\eta)=W_0$ given by

$$W_0^3+[(a_0-a)/a_1\lambda_0]W_0^2+(S_0^2/a_1\lambda_0^3)=0 \quad (4.20)$$

which leads to already known solution (3.14) of (1.1) with $a_2=0$.

In the generic case, (4.19) actually represents three different subcases.

(i) For $S_0=0$ and $a=a_0$, (4.19) is the standard Painlevé equation PXVIII [7]. Its first integral is

$$\dot{W}^2=4W(C+W^2) \quad (4.21)$$

where C is a constant having a physical meaning which depends on the model under consideration; usually it is related to the energy of the system.

We note that equation (4.19) is invariant under the transformation $\lambda_0=\tilde{\lambda}_0\,e^{i\varphi}$, $\eta=\tilde{\eta}\,e^{i\varphi/2}$, $W=\tilde{W}\,e^{-i\varphi}$ with $\tilde{\lambda}_0$, $\tilde{\eta}$, $\tilde{W}$ and $\varphi\in\mathbb{R}$, λ_0, $\tilde{\lambda}_0$ and φ are constants. We can choose λ_0 and W real as long as $H(\xi)=\lambda_0W(\eta)$ remains positive. The solutions of (4.21) are thus real and so is C. This will also be true for the quintic case.

The relation between $\dot{W}^2$ and W given by (4.21) can be illustrated by the 'phase diagrams' of figures 1(a), 1(b) and 1(c) for $C=0$, $C<0$ and $C>0$, respectively. They explicitly show whether W can be finite or singular as well as positive or negative. The parameters λ_0 and a_1 must be chosen such that $M(\xi)$ is real.

Solving (4.21), we obtain the following solutions for (1.1):

$$\psi=(2/a_1)^{1/2}(z-z_0)^{-1}\,e^{-ia_0t}\,e^{i\chi_0} \qquad C=0,\ a_1>0 \quad (4.22)$$

$$\psi=(1/a_1)^{1/2}c_1\,\mathrm{cn}^{-1}(c_1z+c_2,1/\sqrt{2})\,e^{-ia_0t}\,e^{i\chi_0} \qquad C<0,\ a_1>0 \quad (4.23)$$

$$\psi=(-1/a_1)^{1/2}c_1\,\mathrm{cn}(c_1z+c_2,1/\sqrt{2})\,e^{-ia_0t}\,e^{i\chi_0} \qquad C<0,\ a_1<0 \quad (4.24)$$

$$\psi=(2/a_1)^{1/2}c_1\,\mathrm{tn}(c_1z+c_2,1/\sqrt{2})\,\mathrm{dn}(c_1z+c_2,1/\sqrt{2})\,e^{-ia_0t}\,e^{i\chi_0} \qquad C>0,\ a_1>0 \quad (4.25)$$

where c_1 and c_2 are real constants, and C is expressed in terms of c_1 and c_2.

Solution (4.22) is non-periodic and singular at $z=z_0$. Solutions (4.23) and (4.25) are also singular but periodic. Solution (4.24) is a finite and periodic solution, satisfying $0\leqslant|\psi|\leqslant(-2/a_1)^{1/2}|c_1|$.

(ii) For $S_0=0$ and $a\neq a_0$, (4.19) is reduced to the standard Painlevé equation PXIX by the choice

$$a=a_0-\tfrac{1}{2}\lambda_0a_1. \quad (4.26)$$

The first integral is

$$\dot{W}^2=4W(C+W+W^2). \quad (4.27)$$

The form of the solution depends on the value of C. For $C=0$ or $C=\frac{1}{4}$, the polynomial on the right-hand side of (4.27) has a double root.

For $C=0$, we obtain

$$\psi=(2/a_1)^{1/2}c_1\,\mathrm{cosech}(c_1z+c_2)\exp[-i(a_0-c_1^2)t]\exp(i\chi_0) \qquad a_1>0 \quad (4.28)$$

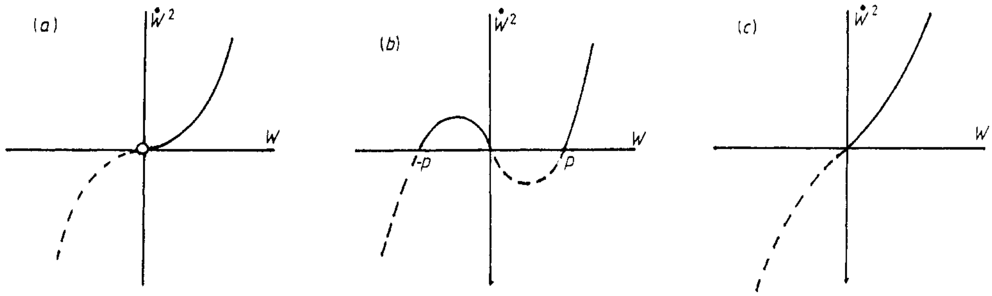

Figure 1. Phase diagrams for equation (4.21). (a) $C=0$ (the circle on this and all subsequent figures denotes a multiple zero of $\dot{W}^2$ or W_x^2). (b) $C=-p^2<0$. (c) $C>0$.

$$\psi=(-2/a_1)^{1/2}c_1\,\mathrm{sech}(c_1z+c_2)\exp[-\mathrm{i}(a_0-c_1^2)t]\exp(\mathrm{i}\chi_0)\qquad a_1<0 \tag{4.29}$$

$$\psi=(2/a_1)^{1/2}c_1\sec(c_1z+c_2)\exp[-\mathrm{i}(a_0+c_1^2)t]\exp(\mathrm{i}\chi_0)\qquad a_1>0 \tag{4.30}$$

with the phase diagrams in figures 2(a) and 2(b). Solution (4.28) is non-periodic and singular. Solution (4.29) is non-periodic and finite. It represents the famous soliton solution of the (1+1)-dimensional integrable NLS equation. Solution (4.30) is periodic but singular (see figure 2(b)).

For $C=\frac{1}{4}$, we obtain similar solutions associated with the phase diagrams of figures 2(c) and 2(d):

$$\psi=(2/a_1)^{1/2}c_1\tan(c_1z+c_2)\exp[-\mathrm{i}(a_0-2c_1^2)t]\exp(\mathrm{i}\chi_0)\qquad a_1>0 \tag{4.31}$$

$$\psi=(2/a_1)^{1/2}c_1\tanh(c_1z+c_2)\exp[-\mathrm{i}(a_0+2c_1^2)t]\exp(\mathrm{i}\chi_0)\qquad a_1>0 \tag{4.32}$$

$$\psi=(2/a_1)^{1/2}c_1\coth(c_1z+c_2)\exp[-\mathrm{i}(a_0+2c_1^2)t]\exp(\mathrm{i}\chi_0)\qquad a_1>0. \tag{4.33}$$

Solution (4.31) is periodic and singular. Solution (4.32) is finite and non-periodic and represents another kind of solitary wave, namely a kink. Solution (4.33) is singular and non-periodic.

For $C<0$, we obtain the following solutions (with $a_1<0$ ($\varepsilon=-1$) or $a_1>0$ ($\varepsilon=1$))

$$\psi=c_1(\varepsilon a_1)^{-1/2}[1-\varepsilon/(1-4C)^{1/2}]^{1/2}\,\mathrm{cn}^{-\varepsilon}\left[c_1z+c_2,\left(\frac{1+(1-4C)^{1/2}}{2(1-4C)^{1/2}}\right)^{1/2}\right]$$
$$\times\exp\{-\mathrm{i}[a_0-c_1^2/(1-4C)^{1/2}]t\}\exp(\mathrm{i}\chi_0) \tag{4.34}$$

$$\psi=c_1(\varepsilon a_1)^{-1/2}[1+\varepsilon/(1-4C)^{1/2}]^{1/2}\,\mathrm{cn}^{-\varepsilon}\left[c_1z+c_2,\left(\frac{-1+(1-4C)^{1/2}}{2(1-4C)^{1/2}}\right)^{1/2}\right]$$
$$\times\exp\{-\mathrm{i}[a_0+c_1^2/(1-4C)^{1/2}]t\}\exp(\mathrm{i}\chi_0). \tag{4.35}$$

The solutions (4.34) and (4.35) for $a_1>0$ ($\varepsilon=1$) are singular and periodic. The solutions (4.34) and (4.35) for $a_1<0$ ($\varepsilon=-1$) are regular and periodic. Figures 2(e) and 2(f) show the corresponding phase diagrams ($\dot{W}^2$, W).

For $0<C<\frac{1}{4}$, we obtain

$$\psi=(2/a_1)^{1/2}\left(\frac{1-(1-4C)^{1/2}}{1+(1-4C)^{1/2}}\right)^{1/2}c_1\,\mathrm{tn}(c_1z+c_2,k)\exp(-\mathrm{i}at)\exp(\mathrm{i}\chi_0)\qquad a_1>0 \tag{4.36}$$

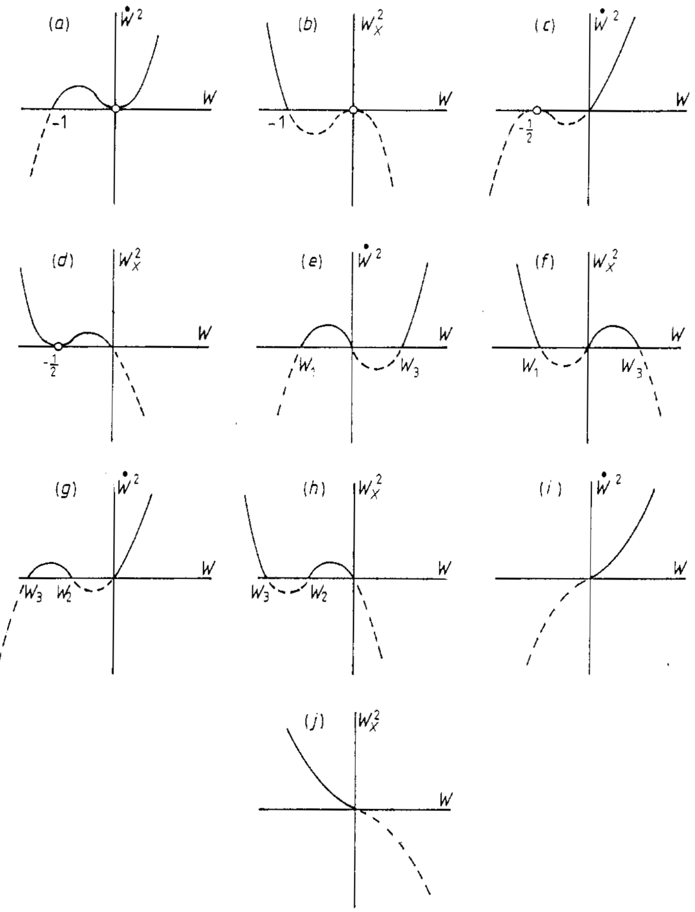

Figure 2. Phase diagrams for equations (4.27). (*a*) $C=0$. (*b*) $C=0$ with $\eta=\mathrm{i}x$. (*c*) $C=\frac{1}{4}$. (*d*) $C=\frac{1}{4}$ with $\eta=\mathrm{i}x$. (*e*) $C<0$, $W_{3,1}=-\frac{1}{2}\pm\frac{1}{2}(1-4C)^{1/2}$. (*f*) $C<0$, $W_{3,1}=-\frac{1}{2}\pm\frac{1}{2}(1-4C)^{1/2}$ and $\eta=\mathrm{i}x$. (*g*) $0<C<\frac{1}{4}$, $W_{2,3}=-\frac{1}{2}\pm\frac{1}{2}(1-4C)^{1/2}$. (*h*) $0<C<\frac{1}{4}$, $W_{2,3}=-\frac{1}{2}\pm\frac{1}{2}(1-4C)^{1/2}$ and $\eta=\mathrm{i}x$. (*i*) $C>\frac{1}{4}$; the curve may have two more critical points (not shown). (*j*) $C>\frac{1}{4}$ and $\eta=\mathrm{i}x$; the curve may have two more critical points (not shown).

$$\psi=(-2/a_1)^{1/2}[\tfrac{1}{2}+\tfrac{1}{2}(1-4C)^{1/2}]^{-1/2}c_1[\tfrac{1}{2}-\tfrac{1}{2}(1-4C)^{1/2} \\ +(1-4C)^{1/2}\,\mathrm{cn}^2(c_1z+c_2,k)]^{1/2}\exp(-\mathrm{i}at)\exp(\mathrm{i}\chi_0) \qquad a_1<0 \qquad (4.37)$$

where

$$k^2=2(1-4C)^{1/2}/[1+(1-4C)^{1/2}] \qquad (4.38a)$$

$$a=a_0-2c_1^2/[1+(1-4C)^{1/2}]. \qquad (4.38b)$$

Also

$$\psi=\left(\frac{2}{a_1}\right)^{1/2}\left(\frac{1-(1-4C)^{1/2}}{1+(1-4C)^{1/2}}\right)^{1/2}c_1\,\mathrm{sn}(c_1z+c_2,k)\exp(-\mathrm{i}at)\exp(\mathrm{i}\chi_0) \qquad a_1>0 \qquad (4.39)$$

$$\psi=(2/a_1)^{1/2}[1+(1-4C)^{1/2}]^{1/2}c_1\,\mathrm{cn}^{-1}(c_1z+c_2,k) \\ \times\{1+(1-4C)^{1/2}+[-1+(1-4C)^{1/2}] \\ \times\mathrm{sn}^2(c_1z+c_2,k)\}^{1/2}\exp(-\mathrm{i}at)\exp(\mathrm{i}\chi_0) \qquad a_1>0 \qquad (4.40)$$

where

$$k^2=[1-(1-4C)^{1/2}]/[1+(1-4C)^{1/2}] \tag{4.41a}$$

$$a=a_0+2c_1^2/[1+(1-4C)^{1/2}]. \tag{4.41b}$$

Solution (4.36) is singular and periodic. Solution (4.37) is regular and periodic. The corresponding diagram is shown in figure 2(g). Solution (4.39) is regular and periodic and solution (4.40) is singular and periodic (see figure 2(h)).

Finally, for $C>\frac{1}{4}$, we have the two solutions

$$\psi=\left(\frac{2}{a_1}\right)^{1/2} c_1 \operatorname{tn}\left[c_1 z+c_2, \left(\frac{2C^{1/2}+\varepsilon}{4C^{1/2}}\right)^{1/2}\right] \exp\{\mathrm{i}[-(a_0+\varepsilon c_1^2/C^{1/2})t+\chi_0]\} \tag{4.42a}$$

$$a_1>0 \qquad \varepsilon=\pm 1. \tag{4.42b}$$

Both solutions are singular and correspond to figure 2(i) for $\varepsilon=-1$ and figure 2(j) for $\varepsilon=+1$.

This completes the solutions of (1, 1) obtained from (4.27).

(iii) For $S_0\neq 0$, relation (4.19) is reduced to the standard Painlevé equation PXXXIII by choosing

$$S_0^2=-\tfrac{1}{8}a_1\lambda_0^3. \tag{4.43}$$

The first integral is

$$\begin{aligned}\dot W^2&=4W^3+2\alpha W^2+4CW+1\\ &=4(W-W_1)(W-W_2)(W-W_3)\equiv P(W)\end{aligned} \tag{4.44}$$

where

$$\alpha=4(a_0-a)/a_1\lambda_0. \tag{4.45}$$

Relation (4.43) implies that $a_1\lambda_0<0$; thus, according to (4.6b), η_0 can be chosen such that η is pure imaginary.

The polynomial $P(W)$ has one triple real root $W_1=-(1/4)^{1/3}$ for

$$C=3(1/4)^{2/3} \tag{4.46a}$$

$$\alpha=6(1/4)^{1/3}. \tag{4.46b}$$

This leads to the phase diagram in figure 3(a) ($\eta=\mathrm{i}x$). Integrating (4.44) and substituting into (3.13b) for χ, we obtain

$$\begin{aligned}\psi=&\{[2/a_1(z-z_0)^2]-\lambda_0(1/4)^{1/3}\}^{1/2}\exp(\mathrm{i}\chi)\\ &\times\exp\{-\mathrm{i}[a_0-\tfrac{3}{2}(1/4)^{1/3}a_1\lambda_0]t\} \qquad a_1>0,\ \lambda_0<0\end{aligned} \tag{4.47a}$$

$$\begin{aligned}\chi=&(\tfrac{1}{8}\lambda_0 a_1)^{1/2}4^{1/3}\{z_0-z+(-2\times 4^{1/3}/\lambda_0 a_1)^{1/2}\\ &\times\tan^{-1}[-\lambda_0 a_1/(2\times 4^{1/3})]^{1/2}(z-z_0)\}.\end{aligned} \tag{4.47b}$$

The solution is obviously non-periodic and singular at $z=z_0$.

The polynomial $P(W)$ has one double real root $W_1=W_2$ and one single root for

$$C=(2W_1^3-1)/2W_1 \tag{4.48a}$$

$$\alpha=-2(2W_1-1/4W_1^2) \tag{4.48b}$$

where W_1 satisfies one of the following conditions:

$$-(1/4)^{1/3}<W_1<0 \tag{4.49a}$$

$$W_1 > 0 \tag{4.49b}$$

$$W_1 < -(1/4)^{1/3}. \tag{4.50}$$

The corresponding solutions of (1.1) are

$$\psi = \lambda_0^{1/2}\{W_1 - [(1+4W_1^3)/4W_1^2]\sec^2\tau\}^{1/2}\exp(\mathrm{i}\chi)\exp(-\mathrm{i}at) \qquad \lambda_0<0,\ a_1>0$$
$$\chi = \frac{4W_1^3}{(1+4W_1^3)^{1/2}}\int\frac{\mathrm{d}\tau}{4W_1^3-(1+4W_1^3)\sec^2\tau}+\chi_0 \tag{4.51}$$
$$\tau = [(-\lambda_0 a_1/2)(4W_1^3+1)/4W_1^2]^{1/2}(z-z_0) \qquad -(1/4)^{1/3} < W_1 < 0 \text{ or } W_1 > 0$$

$$\psi = \lambda_0^{1/2}[W_1 - [(1+4W_1^3)/4W_1^2]\,\mathrm{sech}^2\,\tau]^{1/2}\exp(\mathrm{i}\chi)\exp(-\mathrm{i}at) \qquad \lambda_0<0,\ a_1>0$$
$$\chi = \frac{4W_1^3}{(-1-4W_1^3)^{1/2}}\int\frac{\mathrm{d}\tau}{4W_1^3-(1+4W_1^3)\,\mathrm{sech}^2\,\tau}+\chi_0 \tag{4.52}$$
$$\tau = [(\lambda_0 a_1/2)(4W_1^3+1)/4W_1^2]^{1/2}(z-z_0) \qquad W_1 < -(1/4)^{1/3}$$

$$\psi = \lambda_0^{1/2}\{W_1 + [(1+4W_1^3)/4W_1^2]\,\mathrm{cosech}^2\,\tau\}^{1/2}\exp(\mathrm{i}\chi)\exp(-\mathrm{i}at) \qquad \lambda_0<0,\ a_1>0$$
$$\chi = \frac{4W_1^3}{(-1-4W_1^3)^{1/2}}\int\frac{\mathrm{d}\tau}{4W_1^3+(1+4W_1^3)\,\mathrm{cosech}^2\,\tau}+\chi_0 \tag{4.53}$$

with τ and W_1 as in (4.52). The only finite solution is (4.52). The corresponding phase diagrams are shown in figures (3b) and 3(c).

The polynomial $P(W)$ in (4.44) has three real distinct roots W_1, W_2 and $W_3 = -1/4W_1W_2$ for

$$C = W_1W_2 - (W_1+W_2)/4W_1W_2 \tag{4.54a}$$

$$\alpha = -2(W_1+W_2-1/4W_1W_2) \tag{4.54b}$$

where W_1 and W_2 satisfy

$$W_1 > W_2 > 0 \tag{4.55}$$

or

$$W_2 < W_1 < 0 \qquad W_1 > -1/4W_2^2. \tag{4.56}$$

The two finite solutions of (1.1) for which $W_2 \leqslant W \leqslant W_1$ are given by (see figure 3(d))

$$\psi = (\lambda_0 W)^{1/2}\,\mathrm{e}^{-\mathrm{i}at}\,\mathrm{e}^{\mathrm{i}\chi} \qquad \begin{cases}\lambda_0>0,\ a_1<0,\ W_1>W_2>0 & (4.57a)\\ \lambda_0<0,\ a_1>0,\ W_2<W_1<0,\ W_1>-1/4W_2^2 & (4.57b)\end{cases}$$

$$W = W_1\,\mathrm{cn}^2(\tau,k) + W_2\,\mathrm{sn}^2(\tau,k)$$

$$\chi = \tfrac{1}{2}(W_1 + 1/4W_1W_2)^{-1/2}\int\frac{\mathrm{d}\tau}{W}+\chi_0$$

$$\tau = (W_1+1/4W_1W_2)^{1/2}(-\tfrac{1}{2}\lambda_0 a_1)^{1/2}(z-z_0) \qquad k^2 = 4W_1W_2(W_1-W_2)/(4W_1^2W_2+1).$$

The two singular solutions where $-\infty < W \leqslant W_3$ (see figure 3(d)) are as in (4.57) but with

$$W = -[1+4W_1W_2^2\,\mathrm{sn}^2(\tau,k)]/4W_1W_2\,\mathrm{cn}^2(\tau,k) \qquad \lambda_0<0,\ a_1>0 \tag{4.58}$$

and W_1, W_2 satisfying (4.55) or (4.56).

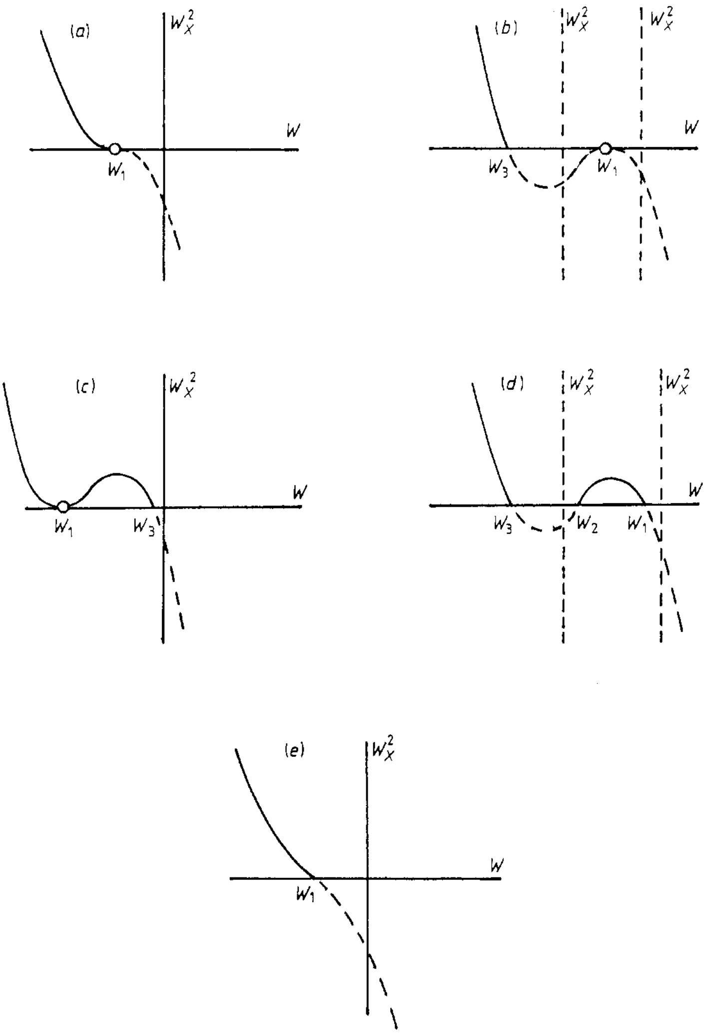

Figure 3. Phase diagram for equation (4.44) with $\eta = \mathrm{i}x$. (a) C and α given by (4.46); $W_1 = -(\frac{1}{4})^{1/3}$. ($b$) C and α given by (4.48); W_1 satisfies (4.49) and $W_3 = -1/4W_1^2$. (c) C and α given by (4.48); W_1 satisfies (4.50) and $W_3 = -1/4W_1^2$. (d) C and α given by (4.54); W_1 and W_2 satisfy (4.55) or (4.56) and $W_3 = -1/4W_1W_2$. (e) C and α as in (4.59); $W_1 < 0$; the curve may have two more critical points (not shown).

Finally, $P(W)$ can have one real root W_1 and two complex conjugate roots $W_2 = W_3^* = p + \mathrm{i}q$, $q > 0$. The solution of (1.1) is then

$$\psi = (\lambda_0 W)^{1/2} \exp[\![\mathrm{i}\{[-a_0 - \tfrac{1}{2}a_1\lambda_0(W_1 + 2p)]t + \chi\}]\!] \qquad \lambda_0 < 0,\ a_1 > 0$$

$$W = [W_1 - A + (W_1 + A)\,\mathrm{cn}(\tau, k)]/[1 + \mathrm{cn}(\tau, k)] \qquad k^2 = (A - p + W_1)/2A \qquad (4.59)$$

$$\tau = 2\sqrt{A}(-\lambda_0 a_1/2)^{1/2}(z - z_0) \qquad A^2 = W_1^2 - 2pW_1 - 1/4W_1$$

$$\chi = \frac{1}{4\sqrt{A}} \int \frac{\mathrm{d}\tau}{W} + \chi_0 \qquad C = 2pW_1 - 1/4W_1 \qquad \alpha = -2(W_1 + 2p) \qquad W_1 < 0.$$

This solution is periodic and singular (see figure 3(e)).

4.4. Solutions of the quintic NLSE *($a_2 \neq 0$)*

As stated above, (4.3) with $a_2 \neq 0$ is not of Painlevé type. We shall thus concentrate on (4.4). To transform it into a standard form we set

$$\lambda(z)=\lambda_0 \qquad \eta=\varepsilon(4a_2/3)^{1/2}\lambda_0 z+\eta_0 \tag{4.60}$$

in (4.5) and obtain the standard Painlevé equation PXXX for $W(\eta)$:

$$\ddot{W}=(1/2W)\dot{W}^2+\tfrac{3}{2}W^3+\tfrac{3}{2}(a_1/a_2\lambda_0)W^2+\tfrac{3}{2}[(a_0-a)/a_2\lambda_0^2]W$$
$$+\tfrac{3}{2}(S_0^2/a_2\lambda_0^4)(1/W). \tag{4.61}$$

This equation admits constant solutions which yield the already known solution (3.14) for (1.1).

For $\dot{W}\neq 0$, the first integral of (4.61) is

$$\begin{aligned}\dot{W}^2&=W^4+\alpha W^3+\beta W^2+4CW-3S_0^2/a_2\lambda_0^4\\&\equiv(W-W_1)(W-W_2)(W-W_3)(W-W_4)\equiv P(W)\end{aligned} \tag{4.62a}$$

where

$$\alpha=\tfrac{3}{2}a_1/a_2\lambda_0 \qquad \beta=3(a_0-a)/a_2\lambda_0^2 \tag{4.62b}$$

and C can still be taken real.

One real quadruple root in $P(W)$ leads to a real solution W only for $S_0=C=\alpha=\beta=0$. We then obtain

$$\psi=(3/4a_2)^{1/4}[\varepsilon/(z-z_0)]^{1/2}\exp(-\mathrm{i}a_0t)\exp(\mathrm{i}\chi_0) \qquad a_2>0,\ a_1=0,\ \varepsilon=\pm1 \tag{4.63}$$

with the phase diagram in figure 4(a).

$P(W)$ has one real triple root W_1 and one real simple root $W_4\neq W_1$ for

$$\beta=-3(2W_1^2+\alpha W_1) \tag{4.64a}$$

$$4C=8W_1^3+3\alpha W_1^2 \tag{4.64b}$$

$$S_0^2=a_2\lambda_0^4W_1^3(W_1+\tfrac{1}{3}\alpha)\geq 0 \tag{4.64c}$$

with

$$W_4=-3W_1-\alpha. \tag{4.64d}$$

This leads to the solution

$$\psi=\left(\lambda_0\frac{3W_4-(W_4-W_1)^2a_2\lambda_0^2(z-z_0)^2W_1}{3-(W_4-W_1)^2a_2\lambda_0^2(z-z_0)^2}\right)^{1/2}\exp(\mathrm{i}\chi)\exp[-\tfrac{1}{3}\mathrm{i}(a_2\lambda_0^2\beta-3a_0)t]$$

$$\chi=\frac{S_0}{\lambda_0}\left[\frac{z-z_0}{W_1}+\frac{1}{(-3W_4)^{1/2}}\frac{W_4+W_1(W_4-W_1)^2a_2\lambda_0^2}{[(W_4-W_1)^2a_2\lambda_0^2W_1]^{3/2}}\right.$$
$$\left.\times\tan^{-1}\left(\frac{(z-z_0)[-3W_4(W_4-W_1)^2a_2\lambda_0^2]^{1/2}}{3W_4}\right)\right]+\chi_0 \tag{4.65}$$

where β, S_0 and W_4 are as in (4.64) and

$a_2>0$ if $W_4\leq 0\leq W_1$, $W_4\neq W_1$ (see figure 4(b))

$a_2<0$, $\lambda_0>0$ if $0\leq W_1<W_4$ or $0\leq W_4<W_1$ (see figure 4(c)).

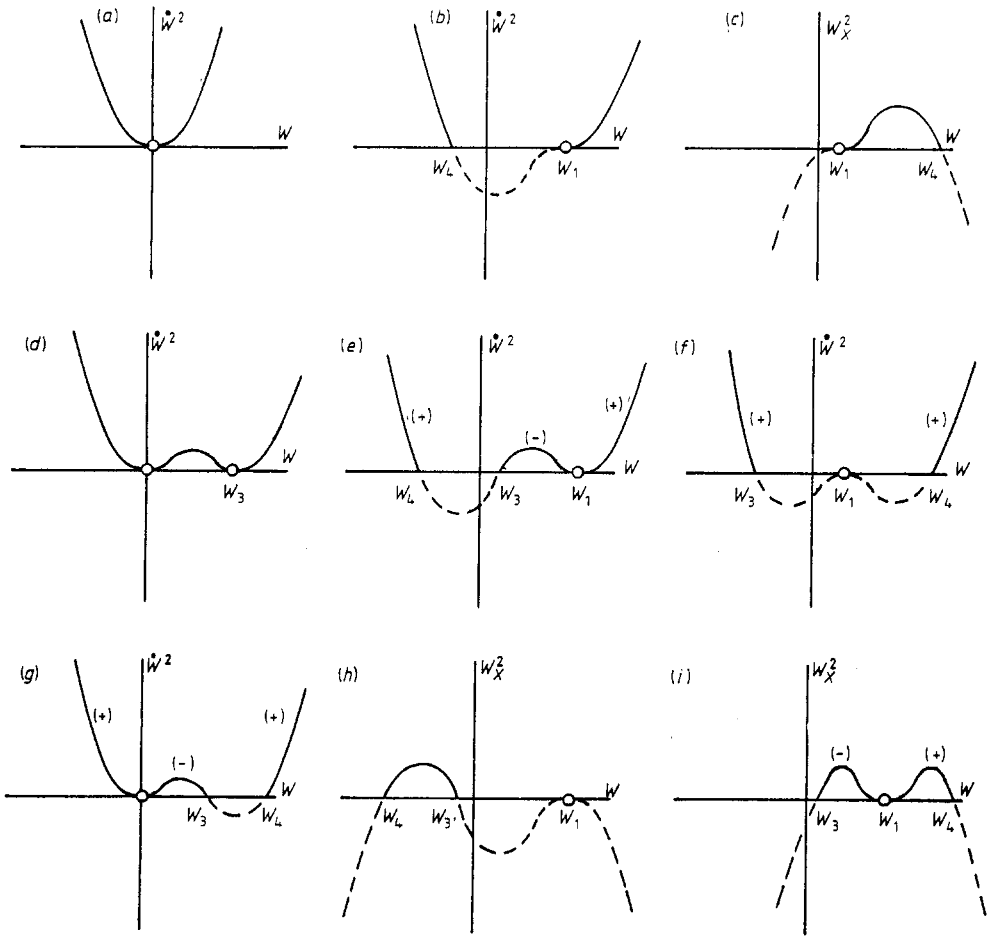

Figure 4. Phase diagram for equation (4.62a). (a) $S_0 = C = \alpha = \beta = 0$; only the case $W > 0$ is pertinent. (b) β, C, S_0, W_4 as in (4.64) and $W_4 \leqslant 0 \leqslant W_1$, $W_4 \neq W_1$. (c) β, C, S_0, W_4 as in (4.64) and $0 \leqslant W_1 < W_4$ or $0 \leqslant W_4 < W_1$ (not shown). (d) β, C, S_0 as in (4.66); $W_3 = -\frac{1}{2}\alpha$. (e), (f), (g) β, C, S_0 and W_4 as in (4.68); $a_2 > 0$. (h) (i) (j) β, C, S_0 and W_4 as in (4.68); $a_2 < 0$ ($\eta = \mathrm{i}x$). (k) β, C, S_0 and W_4 as in (4.71). (l) β, C, S_0 and W_4 as in (4.71), $a_2 < 0$ ($\eta = \mathrm{i}x$). (m) β, C, S_0 and p as in (4.74), $W_2 \leqslant 0 \leqslant W_1$, $W_2 \neq W_1$. (n) β, C, S_0 and p as in (4.74), $a_2 < 0$ ($\eta = \mathrm{i}x$), $0 \leqslant W_2 < W_1$.

The only finite solutions are those for $a_2 < 0$. The solution for $a_2 > 0$ actually represents two solutions. One with $\lambda_0 > 0$ for $(W_4 - W_1)^2 a_2 \lambda_0^2 (z - z_0)^2 > 3$ and another with $\lambda_0 < 0$ for $(W_4 - W_1)^2 a_2 \lambda_0^2 (z - z_0)^2 < 3$.

Equation (4.62) also leads to real solutions for two double real roots $W_1 = 0$ and $W_3 = -\frac{1}{2}\alpha$ when

$$C = S_0 = 0 \qquad \beta = 9a_1^2/16a_2^2\lambda_0^2. \tag{4.66}$$

The solution of (1.1) then is

$$\psi = [\![-\tfrac{3}{4}(a_1/a_2)\{1 + \varepsilon \exp[\tfrac{1}{2}(3/a_2)^{1/2} a_1 (z - z_0)]\}^{-1}]\!]^{1/2} \times \exp\{\mathrm{i}[(-a_0 + 3a_1^2/16a_2)t + \chi_0]\} \qquad a_2 > 0,\ \varepsilon = \pm 1. \tag{4.67}$$

The case $\varepsilon = 1$ and $a_1 < 0$ gives a finite non-periodic solution while the cases $\varepsilon = -1$, $a_1 < 0$ or $a_1 > 0$ lead to two singular solutions (see figure 4(d)). The norm of the finite solution has the form of a kink or an antikink.

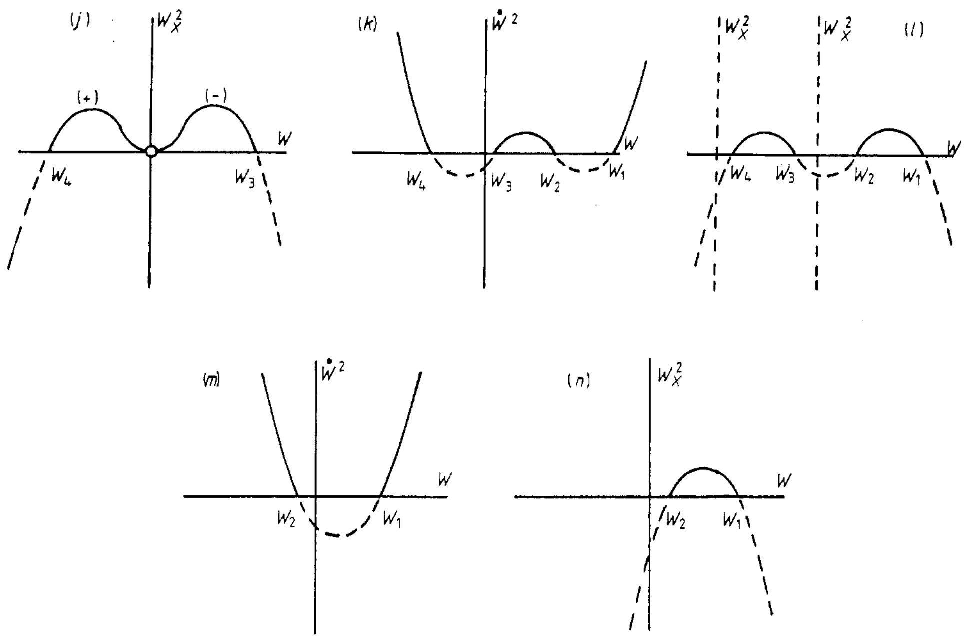

Figure 4. (continued).

The polynomial $P(W)$ can have one double real root $W_1=W_2\neq W_3\neq W_4\neq W_1$ when

$$\beta=[W_1^2+2W_1(W_3+W_4)+W_3W_4] \tag{4.68a}$$

$$4C=-W_1^2(W_3+W_4)-2W_1W_3W_4 \tag{4.68b}$$

$$3S_0^2=-a_2\lambda_0^4W_1^2W_3W_4\geqslant 0 \tag{4.68c}$$

where

$$W_4=-2W_1-W_3-\alpha. \tag{4.68d}$$

Relations (4.68) impose one of the following possibilities:

$$W_1>0 \qquad W_3W_4\geqslant 0 \qquad a_2<0 \tag{4.69a}$$

$$W_1>0 \qquad W_3W_4\leqslant 0 \qquad a_2>0 \tag{4.69b}$$

$$W_1=0\Rightarrow W_4=-W_3-\alpha. \tag{4.69c}$$

The solutions of (1.1) in this case are

$$\begin{aligned}
&\psi=(\lambda_0W)^{1/2}\,\mathrm{e}^{-\mathrm{i}at}\,\mathrm{e}^{\mathrm{i}\chi} \qquad W=(\alpha+\beta\cosh\tau)/(\gamma+\delta\cosh\tau)\\
&\chi=\left(\frac{3}{4a_2}\right)^{1/2}\frac{S_0}{\lambda_0^2[(W_1-W_3)(W_1-W_4)]^{1/2}}\\
&\qquad\times\left(\frac{\delta}{\beta}\tau-\frac{\alpha\delta-\beta\gamma}{\beta}\int\frac{\mathrm{d}\tau}{\alpha+\beta\cosh\tau}\right)+\chi_0\\
&\tau=[(W_1-W_3)(W_1-W_4)]^{1/2}(\tfrac{4}{3}a_2)^{1/2}\lambda_0(z-z_0)\\
&\alpha=-2W_3W_4+W_1(W_3+W_4) \qquad \beta=\varepsilon W_1(W_4-W_3)\\
&\gamma=2W_1-(W_3+W_4) \qquad \delta=\varepsilon(W_4-W_3)
\end{aligned} \tag{4.70}$$

where a, S_0 and W_4 are given by (4.62b), (4.68) and W_1, W_3, W_4 can be ordered according to figures 4(e)-4(j).

All the possible phase diagrams are given in these figures. The first three are for $a_2>0$ and the last three are for $a_2<0$. The sign in brackets corresponds to ε. When $W>0$, we choose $\lambda_0>0$ and vice versa. This shows that (4.70) represents in fact thirteen different solutions. The particularly interesting cases are the finite ones. Those corresponding to $W_3\leqslant W\leqslant W_1$ in figure 4(e), $0\leqslant W\leqslant W_3$ in figure 4(g), $W_3\leqslant W\leqslant W_1$ or $W_1\leqslant W\leqslant W_4$ in figure 4(i) and $W_4\leqslant W\leqslant 0$ or $0\leqslant W\leqslant W_3$ in figure 4(j) represent solitary waves. The solution corresponding to $W_4\leqslant W\leqslant W_3$ in figure 4(h) is trigonometrically periodic (since τ is imaginary).

The polynomial $P(W)$ can have four real distinct ordered roots $W_1>W_2>W_3>W_4$ for

$$\beta = W_1W_2+(W_1+W_2)(W_3+W_4)+W_3W_4 \tag{4.71a}$$

$$4C=-W_1W_2(W_3+W_4)-W_3W_4(W_1+W_2) \tag{4.71b}$$

$$3S_0^2=-a_2\lambda_0^4W_1W_2W_3W_4\geqslant 0 \tag{4.71c}$$

where

$$W_4=-W_1-W_2-W_3-\alpha. \tag{4.71d}$$

For $W_1>W_2>W_3\geqslant 0\geqslant W_4$, $W_3\neq W_4$ and $a_2>0$, we obtain the singular solution

$$\begin{aligned}
&\psi=(\lambda_0W)^{1/2}\,\mathrm{e}^{-\mathrm{i}at}\,e^{\mathrm{i}\chi} \qquad a_2>0\\
&W=[W_4(W_1-W_3)-W_3(W_1-W_4)\,\mathrm{sn}^2(\tau,k)]/[W_1-W_3-(W_1-W_4)\,\mathrm{sn}^2(\tau,k)]\\
&\chi=\frac{S_0}{[(W_1-W_3)(W_2-W_4)]^{1/2}}\frac{1}{\lambda_0^2}\left(\frac{3}{a_2}\right)^{1/2}\int\frac{\mathrm{d}\tau}{W}+\chi_0\\
&k^2=[(W_2-W_3)(W_1-W_4)]/[(W_1-W_3)(W_2-W_4)]\\
&\tau=[(W_1-W_3)(W_2-W_4)]^{1/2}(\tfrac{1}{3}a_2)^{1/2}\lambda_0(z-z_0).
\end{aligned} \tag{4.72}$$

For $W>0$, we take $\lambda_0>0$, and vice versa (see figure 4(k)). A regular solution is given by the change $W_4\leftrightarrow W_3$, $W_1\leftrightarrow W_2$ in W and is always positive (see figure 4(k)). Thus we take $\lambda_0>0$.

For $W_1>W_2>W_3>W_4\geqslant 0$ and $a_2<0$, we have the regular solution

$$\begin{aligned}
W=[W_2(W_1-W_3)-W_3(W_1-W_2)\,&\mathrm{sn}^2(\tau,k)]/[W_1-W_3\\
&-(W_1-W_2)\,\mathrm{sn}^2(\tau,k)] \qquad a_2<0,\ \lambda_0>0
\end{aligned} \tag{4.73}$$

$$k^2=(W_1-W_2)(W_3-W_4)/(W_1-W_3)(W_2-W_4)$$

with χ and τ as in (4.72) with $a_2\to -a_2$. Another regular solution is obtained by the change $W_2\leftrightarrow W_4$, $W_1\leftrightarrow W_3$. The phase diagrams are shown in figure 4(l).

For $W_1>W_2\geqslant 0\geqslant W_4$, $W_2\neq W_3$ and $a_2<0$, we obtain a regular positive solution in the same form as (4.73). Also, by the change $W_2\leftrightarrow W_4$, $W_1\leftrightarrow W_3$, we generate another regular negative solution for W; thus we take $\lambda_0<0$ (see figure 4(l)).

Finally, $P(W)$ can have two distinct real roots W_1, W_2 and two complex conjugate roots $W_3=W_4^*=p+\mathrm{i}q$, $q>0$ for

$$\beta=W_1W_2+(W_1+W_2)2p+p^2+q^2 \tag{4.74a}$$

$$4C=-W_1W_22p+(p^2+q^2)(W_1+W_2) \tag{4.74b}$$

$$3S_0^2=-a_2\lambda_0^4W_1W_2(p^2+q^2)\geqslant 0 \tag{4.74c}$$

where

$$2p = -W_1 - W_2 - \alpha. \tag{4.74d}$$

The corresponding solutions for (1.1) are then

$$\begin{aligned}
&\psi = (\lambda_0 W)^{1/2} e^{-i a t} e^{i\chi} \\
&W = [W_1 B - \varepsilon W_2 A + (W_2 A + \varepsilon W_1 B)\,\mathrm{cn}(\tau, k)]/[B - \varepsilon A + (A + \varepsilon B)\,\mathrm{cn}(\tau, K)] \\
&\chi = (-\varepsilon W_1 W_2 / AB)^{1/2} \tfrac{1}{2}(p^2 + q^2)^{1/2} \int \mathrm{d}\tau / W \\
&\tau = (AB)^{1/2} 2(\tfrac{1}{3}\varepsilon a_2)^{1/2} \lambda_0 (z - z_0) \qquad k^2 = \varepsilon[(A + \varepsilon B)^2 - (W_1 - W_2)^2]/4AB \\
&A^2 = (W_1 - p)^2 + q^2 \qquad B^2 = (W_2 - p)^2 + q^2
\end{aligned} \tag{4.75}$$

where a, S_0, p are given by (4.62b), (4.74) and $\varepsilon = 1$ when $a_2 > 0$, $\varepsilon = -1$ when $a_2 < 0$. For the case $a_2 > 0$, solution (4.75) actually represents two different solutions; in the region where $W > 0$, we choose $\lambda_0 > 0$ and vice versa (see figure 4(m)). The case $a_2 < 0$ is regular and we choose $\lambda_0 > 0$ if we have $0 < W_2 \leqslant W \leqslant W_1$ (see figure 4(h)), or $\lambda_0 < 0$ if we have $W_2 \leqslant W \leqslant W_1 < 0$.

5. Solutions invariant under a subgroup involving dilations

Solutions involving some symmetry under dilations are usually called similarity solutions. Among all the subalgebras listed in § 3 and involving dilations, only one leads to second-order ODE for $M(\xi)$ which is of Painlevé type under certain conditions. This subalgebra is

$$(d + a j_3,\, t,\, p_3) \qquad a \geqslant 0 \tag{5.1}$$

which leads to relations (3.23a)-(3.23d) (with $b = 0$).

Submitting (3.23d) to the Painlevé test, we find that (3.23d) passes the test and can be reduced to standard Painlevé type equations only if $a = 0$. However, in the case $a \neq 0$ and $S_0 = 0$, we can reduce the order of the equation by one, since it is invariant under the change $\xi \to \xi + \xi_0$, ξ_0 is a constant. The transformation to use is

$$y = M \qquad w(y) = \xi \qquad w_y = h \tag{5.2}$$

which leads to

$$-(a^2 + 1)h_y - 2ah^2 + (y - a_1 y^3)h^3 = 0 \qquad a_2 = 0 \tag{5.3}$$

$$-(a^2 + 1)h_y - ah^2 + (\tfrac{1}{4}y - a_2 y^5)h^3 = 0 \qquad a_1 = 0. \tag{5.4}$$

These are Abel type equations that are difficult to integrate. We will not go into them.

Let us come back to the Painlevé type equations (for $a = 0$) and separately analyse the cubic and quintic cases.

5.1. *The cubic case ($a_2 = 0$, $\delta = 1$)*

To transform (3.23d) (with $a = 0$) into standard form we again make the transformation (3.28) and then use relations (4.6a), (4.6b) (with $\xi \equiv \theta$), and obtain for W:

$$\ddot{W} = (1/2W)\dot{W}^2 + 4W^2 - (4/a_1\lambda_0)W + (4S_0^2/a_1\lambda_0^3)(1/W). \tag{5.5}$$

Constant solutions $W = W_0$ lead to already known results (3.23f) and (3.23g).

The generic case is actually very similar to one treated above (see (4.19)). It can be divided into two different cases.

(i) For $S_0=0$, (5.5) is reduced to the standard Painlevé equation PXIX by the substitution

$$\lambda_0=-2/a_1. \tag{5.6}$$

The first integral is (4.27) and $\eta-\eta_0=\varepsilon i\theta$ is pure imaginary. The solutions for (1.1) are still obtained in a similar manner as in § 4.1 and will be simply listed here with the values of some parameters and a reference to the corresponding phase diagram. These diagrams contain all information about the degeneracy of the roots and on the singular, regular, periodic or non-periodic character of $W(\theta)$.

For $C=0$ (see figure 2(b)), we have

$$\psi=(2/a_1)^{1/2}\rho^{-1}\sec(\theta-\theta_0)\exp(-\mathrm{i}a_0t)\exp(\mathrm{i}\chi_0) \qquad a_1>0. \tag{5.7}$$

For $C=\frac{1}{4}$ (see figure 2(d)), we have

$$\psi=(1/a_1)^{1/2}\rho^{-1}\tanh[(1/\sqrt{2})(\theta-\theta_0)]\exp(-\mathrm{i}a_0t)\exp(\mathrm{i}\chi_0) \qquad a_1>0 \tag{5.8}$$

$$\psi=(1/a_1)^{1/2}\rho^{-1}\coth[(1/\sqrt{2})(\theta-\theta_0)]\exp(-\mathrm{i}a_0t)\exp(\mathrm{i}\chi_0) \qquad a_1>0. \tag{5.9}$$

The solution (5.8) is finite and non-periodic as a function of θ, but has a pole for $\rho=0$. Solution (5.9) is singular for $\theta=\theta_0$.

For $C<0$ (see figure 2(f)) we obtain

$$\psi=(\varepsilon/a_1)^{1/2}(c_1^2+\varepsilon)^{1/2}\rho^{-1}\,\mathrm{cn}\{c_1\theta+c_2,[(c_1^2-1)/2c_1^2]^{1/2}\}\exp(-\mathrm{i}a_0t)\exp(\mathrm{i}\chi_0) \tag{5.10}$$

where $|c_1|>1$, $\varepsilon=1$ for $a_1>0$ and $\varepsilon=-1$ for $a_1<0$.

For $0<C<\frac{1}{4}$ (see figure 2(h)) we obtain the finite solution

$$\begin{aligned}\psi=(2/a_1)^{1/2}(1-c_1^2)^{1/2}\rho^{-1}\,&\mathrm{sn}\{c_1\theta+c_2,[(1-c_1^2)/c_1^2]^{1/2}\}\\ &\times\exp(-\mathrm{i}a_0t)\exp(\mathrm{i}\chi_0) \qquad a_1>0, 0<|c_1|<\tfrac{1}{2}\end{aligned} \tag{5.11}$$

and the singular solution

$$\begin{aligned}\psi=(2/a_1)^{1/2}\rho^{-1}&[\![c_1^2+(c_1^2-1)\,\mathrm{sn}^2\{c_1\theta+c_2,[(1-c_1^2)/c_1^2]^{1/2}\}]\!]\\ &\times[\![\mathrm{cn}\{c_1\theta+c_2,[(1-c_1^2)/c_1^2]^{1/2}\}]\!]^{-1}\exp(-\mathrm{i}a_0t)\exp(\mathrm{i}\chi_0).\end{aligned} \tag{5.12}$$

Finally for $C>\frac{1}{4}$ (see figure 2(j)) we have

$$\begin{aligned}\psi=(2/a_1)^{1/2}\rho^{-1}c_1\,&\mathrm{tn}\{c_1\theta+c_2,\tfrac{1}{2}[(2c_1^2+1)/c_1^2]^{1/2}\}\\ &\times\exp(-\mathrm{i}a_0t)\exp(\mathrm{i}\chi_0) \qquad a_1>0, |c_1|>1/\sqrt{2}.\end{aligned} \tag{5.13}$$

(ii) For $S_0\neq 0$, (5.5) is reduced to the standard Painlevé equation PXXXIII by the substitution (4.43). The first integral is (4.44) with

$$\alpha=-4/a_1\lambda_0. \tag{5.14}$$

According to (4.43) and (4.6b), $\eta-\eta_0$ is still pure imaginary here.

The solutions of (1.1) can be classified according to the value of C and α, exactly as in § 4.3.

For C and α given by (4.46a) and (4.46b) we obtain (see figure 3(a))

$$\begin{aligned}\psi=(2/a_1)^{1/2}\rho^{-1}&[\tfrac{1}{3}+1/(\theta-\theta_0)^2]^{1/2}\\ &\times\exp\mathrm{i}[\![\tfrac{2}{9}\{\theta-\theta_0-\sqrt{3}\tan^{-1}[(\theta+\theta_0)/\sqrt{3}]\}-a_0t+\chi_0]\!] \qquad a_1>0.\end{aligned} \tag{5.15}$$

For C and α given by (4.48a) and (4.48b) and W_1 satisfying one of the following conditions:

$$-(1/4)^{1/3} < W_1 < 0 \qquad (5.16a)$$

$$\tfrac{1}{2} > W_1 > 0 \qquad \text{(see figure 3}(b)) \qquad (5.16b)$$

$$W_1 < -(1/4)^{1/3} \qquad \text{(see figure 3}(c)) \qquad (5.17)$$

we obtain the corresponding solutions for (1.1):

$$\begin{aligned}\psi &= (2/a_1)^{1/2}[1/(8W_1^3-1)]^{1/2}\rho^{-1}[4W_1^3-(1+4W_1^3)\sec^2\tau]^{1/2}\\ &\qquad \times \exp(\mathrm{i}\chi)\exp(-\mathrm{i}a_0t) \qquad a_1>0 \qquad (5.18)\\ \chi &= \frac{-4W_1^3}{(4W_1^2+1)^{1/2}}\int\frac{\cos^2\tau\,\mathrm{d}\tau}{1+4W_1^3\sin^2\tau}+\chi_0\\ \tau &= [(4W_1^3+1)/(1-8W_1^3)]^{1/2}(\theta-\theta_0) \qquad 0<W_1<\tfrac{1}{2} \text{ or } -(1/4)^{1/3}<W_1<0\end{aligned}$$

$$\begin{aligned}\psi &= (2/a_1)^{1/2}[1/(8W_1^3-1)]^{1/2}\rho^{-1}[4W_1^3\\ &\qquad -(1+4W_1^3)\,\mathrm{sech}^2\tau]^{1/2}\exp(\mathrm{i}\chi)\exp(-\mathrm{i}a_0t) \qquad a_1>0 \qquad (5.19)\\ \chi &= \frac{4W_1^3}{(-1-4W_1^3)^{1/2}}\int\frac{\mathrm{d}\tau}{4W_1^3-(1+4W_1^3)\,\mathrm{sech}^2\tau}+\chi_0\\ \tau &= [(1+4W_1^3)/(8W_1^3-1)]^{1/2}(\theta-\theta_0) \qquad W_1<-(1/4)^{1/3}\end{aligned}$$

$$\begin{aligned}\psi &= (2/a_1)^{1/2}[1/(8W_1^3-1)]^{1/2}\rho^{-1}\\ &\qquad \times[4W_1^3+(1+4W_1^3)\,\mathrm{cosech}^2\tau]^{1/2}\exp(\mathrm{i}\chi)\exp(-\mathrm{i}a_0t)\\ \chi &= \frac{4W_1^3}{(-1-4W_1^3)^{1/2}}\int\frac{\mathrm{d}\tau}{4W_1^3+(1+4W_1^3)\,\mathrm{cosech}^2\tau}+\chi_0\end{aligned} \qquad (5.20)$$

with τ and W_1 as in (5.19). The solution (5.19) is finite as a function of τ and has a pole for $\rho=0$. The others are singular as functions of τ.

For C and α given by (4.5a) and (4.54b) where W_1 and W_2 satisfy

$$W_1 > W_2 > 0 \qquad 1/4W_1W_2 > W_1 + W_2 \qquad (5.21)$$

or

$$W_2 < W_1 < 0 \qquad W_1 > -1/4W_2^2 \qquad (5.22)$$

we obtain three real ordered roots for $P(W)$. The two finite solutions of (1.1) for which $W_2 \leqslant W \leqslant W_1$ are given by (see figure 3(d))

$$\psi = \left[\frac{2}{a_1}\left(W_1+W_2-\frac{1}{4W_1W_2}\right)W\right]^{1/2}\exp(\mathrm{i}\chi)\exp(-\mathrm{i}a_0t) \qquad \begin{cases}a_1<0 \text{ for } (5.21)\\ a_1>0 \text{ for } (5.22)\end{cases} \qquad (5.23)$$

$$\begin{aligned}W &= W_1\,\mathrm{cn}^2(\tau,k)+W_2\,\mathrm{sn}^2(\tau,k)\\ \chi &= \tfrac{1}{2}(W_1+1/4W_1W_2)^{-1/2}\int\frac{\mathrm{d}\tau}{W}+\chi_0\\ \tau &= (W_1+1/4W_1W_2)^{1/2}[(1/4W_1W_2)-W_1W_2]^{1/2}(\theta-\theta_0)\\ k^2 &= 4W_1W_2(W_1-W_2)/(4W_1^2W_2+1).\end{aligned}$$

The two singular ones where $-\infty < W \leqslant W_3$ are as in (5.23) but with (see figure 3(d))

$$W = -\frac{1+4W_1W_2^2\,\mathrm{sn}(\tau,k)}{4W_1W_2\,\mathrm{cn}^2(\tau,k)} \qquad a_1>0 \tag{5.24}$$

and W_1, W_2 still satisfying (5.21) or (5.22).

Finally, $P(W)$ can have one real root $W_1 < -\frac{1}{2}$ and two complex conjugate roots $W_2 = W_3^* = p + \mathrm{i}q$, $q>0$. The corresponding singular solution (see figure 3(e)) is given by

$$\psi = \{(8/a_1)[W_1/(4W_1^2-1)]W\}^{1/2}\exp(\mathrm{i}\chi)\exp(-\mathrm{i}a_0t) \qquad a_1>0 \tag{5.25}$$

with W, χ, k, A as in (4.59) and

$$\tau = 2\sqrt{A}[4W_1/(1-4W_1^2)]^{1/2}(\theta-\theta_0) \qquad W_1 < -\tfrac{1}{2}.$$

5.2. Quintic case ($a_1 = 0$, $\delta = \frac{1}{2}$)

To transform (3.23d) (with $a=0$) into a standard form we make the substitution (4.60) (with $z \to \theta$) and obtain the standard Painlevé equation PXXX

$$\ddot{W} = (1/2W)\dot{W}^2 + \tfrac{3}{2}W^3 - (3/8a_2\lambda_0^2)W + \tfrac{3}{2}(S_0^2/a_2\lambda_0^4)(1/W). \tag{5.26}$$

This equation admits constant solutions leading to (3.23e). In the generic case the first integral is similar to (4.62a) but with

$P(W)$ has one real triple root $W_1 = (1/\lambda_0)(1/8a_2)^{1/2}$ and one simple root $W_4 = -3W_1$ for

$$C = (8/\lambda_0^3)(1/8a_2)^{1/2} \tag{5.28a}$$

$$S_0^2 = 1/64a_2. \tag{5.28b}$$

This leads to the singular solution

$$\psi = \left(\frac{\varepsilon}{(8a_2)^{1/2}}\,\frac{9+2(\theta-\theta_0)^2}{3-2(\theta-\theta_0)^2}\right)^{1/2}\rho^{-1/2}\exp\left[\!\left[\mathrm{i}\left\{\frac{-\varepsilon}{2\sqrt{2}}\left[-3(\theta-\theta_0)+(4/\sqrt{2})\tan^{-1}\left(\frac{3(\theta-\theta_0)}{\sqrt{2}}\right)\right.\right.\right.\right.$$
$$\left.\left.\left.\left. -a_0t+\chi_0\right]\right\}\right]\!\right] \qquad a_2>0 \tag{5.29}$$

where $\varepsilon=-1$ for $(\theta-\theta_0)^2 > \frac{3}{2}$ and $\varepsilon = 1$ for $(\theta-\theta_0)^2 < \frac{3}{2}$. The corresponding phase diagram is given in figure 4(b).

$P(W)$ has one real double root W_1 under the conditions (4.68) which can be rewritten as

$$\lambda_0^2 = 3/4a_2(3W_1^2 - W_3W_4) > 0 \tag{5.30a}$$

$$4C = 2W_1^3 - 2W_1W_3W_4 \tag{5.30b}$$

$$S_0^2 = \frac{-3}{16a_2}\,\frac{W_1^2W_3W_4}{(3W_1^2-W_3W_4)^2} \geqslant 0 \tag{5.30c}$$

where

$$W_4 = -2W_1 - W_3. \tag{5.30d}$$

Thus all the parameters are given in terms of W_1 and $W_3 \in \mathbb{R}$. Conditions (5.30a), (5.30b) and (5.30d) state that a_2 must be positive and $W_3 W_4 \leq 0$. By ordering the roots as $W_1 > W_3 \geq 0 \geq W_4$, $W_3 \neq W_4$ or as $W_4 > W_1 \geq 0 \geq W_3$, $W_1 \neq W_3$, one obtains the phase diagrams in figures 4(e) and 4(f). The solutions for (1.1) can all be given in the form

$$\psi = (\lambda_0 W)^{1/2} \rho^{-1/2} \exp(\mathrm{i}\chi) \exp(-\mathrm{i} a_0 t) \tag{5.31}$$

where χ, W are as in (4.70) with $z \to \theta$ and λ_0, S_0, W_4 as in (5.30) and $a_2 > 0$. The sign in figures 4(e) and 4(f) refers to ε in the solution. Here again, for $W > 0$ we choose $\lambda_0 > 0$ and vice versa.

When the coefficients in $P(W)$ satisfy (4.71) we have four real ordered roots $W_1 > W_2 > W_3 > W_4$. One can use (4.71) to show that the real solution W is obtained only for $a_2 > 0$. The corresponding phase diagram is the one in figure 4(k). We thus expect one finite solution with $W_3 \leq W \leq W_2$ and one singular solution which will be positive for certain values of θ and negative for others. The singular solution has the form (5.31) where W, χ are as in (4.72) (with $z \to \theta$), and λ_0, S_0, W_4 satisfying (4.71). The regular one is obtained by the permutation $W_3 \leftrightarrow W_4$, $W_1 \leftrightarrow W_2$.

Finally $P(W)$ can have two distinct real roots W_1, W_2 and two complex conjugate ones $W_3 = W_4^* = p + \mathrm{i}q$, $q > 0$ when conditions (4.74) are imposed. Still here, the solutions which correspond to phase diagram in figures 4(m) and 4(n) have the form (5.31) where χ, W are as in (4.75) (with $z \to \theta$). The solution is singular for $a_2 > 0$ ($W_1 \geq 0 \geq W_2$, $W_1 \neq W_2$) and finite for $a_2 < 0$ ($W_1 > W_2 \geq 0$).

An overall comment on the solutions presented in this section is that they are all functions of the azimuthal angle θ in the (x, y) plane. Since these functions do not, in general, have period 2π (or any period at all, in some cases), the solutions will be multivalued. This must be taken into account in any physical interpretation.

6. Conclusions

This paper, the second in a series, is devoted to the presentation of reduced equations and of some group-invariant solutions for the non-linear Schrödinger equation (1.1). Cylindrically and spherically invariant solutions have been presented in separate papers [9, 10]. Here we have concentrated on translationally invariant solutions (§ 4) and on a specific type of dilationally invariant solutions (§ 5).

Each of the numerous solutions presented above actually represents a conjugacy class of solutions. The entire class is obtained by applying a general symmetry group transformation to the representative solution. The allowed transformations are extended Galilei transformations, coordinate reflections and time reversal in all cases, and also dilations, whenever $a_1 = 0$, or $a_2 = 0$. The transformation formulae were presented in the first paper of this series (see [1], equations (2.6)-(2.8)).

One aspect that is well illustrated by the results of this paper is that the combination of group theory and singularity analysis is a powerful tool for obtaining explicit analytic particular solutions of non-linear PDE. From this point of view it is quite crucial to perform a rigorous subgroup classification, rather than to simply choose 'intuitively obvious' subgroups for performing a reduction. As an example, consider the case of cylindrical boundary conditions. Intuitively, one could imagine that the subalgebra (j_3, p_3, t), leading to cylindrically symmetric static solutions, is a good candidate for performing the reduction. The subalgebra analysis, on the other hand, has shown [1] that this algebra is just one member of the family $(j_3+am,\ t+(b-a_0)m,\ p_3)$ of algebras, where the constants a and b have an invariant meaning (i.e. algebras with constants (a, b) and (a', b') are mutually conjugate only if $|a|=|a'|$, $|b|=|b'|$). Painlevé type equations are obtained only for $a^2=\frac{1}{16}$, $a_0-b=3a_1^2/16a_2$ if $a_2\neq 0$, and for $a^2=\frac{1}{9}$ if $a_2=0$ [9]. The corresponding solutions would have been missed, had we set $a=0$, or $b=0$ *ab initio*. Similarly, the algebra $(t-a_0m+ak_3,\ p_1,\ p_2)$ leads to different types of solutions for different values of the constant a.

Two technical restrictions have so far been made. The first is that we have only presented reductions that lead to second-order ODE. Eight of the subgroups with generic orbits of codimension one (when projected into the space of independent variables) lead to third-order equations. They will be discussed in the third paper of this series. It should be noted that Tajiri [15] and Boiti and Pempinelli [16] have already discussed some similarity solutions of the cubic non-linear Schrödinger equation in $(1+1)$ and $(2+1)$ dimensions. In particular, solutions of a reduction to a third-order ODE have been obtained [16] in terms of the fourth Painlevé transcendent. This in itself shows that a careful singularity analysis of the third-order ODE can yield interesting results.

A further restriction made above is that we have only considered subgroups that lead to invariants $I_i(x, y, z, t, \psi, \psi^*)$ that provide a non-singular transformation from $\{\psi, \psi^*\}$ to I_i. This made it possible to always express ψ (and ψ^*) in terms of similarity variables. If the corresponding mapping is not invertible, one is lead to what Ovsiannikov [5] calls 'partially invariant solutions'. We plan to return to these in a later publication.

Physical interpretations and applications of the obtained solutions also remain open. Clearly they depend crucially on the model under consideration, i.e. on the interpretation of the function ψ. As always, the obtained explicit solutions can serve as a basis for a perturbation theory that should provide further approximate solutions. They can also serve as the basis of a quantisation procedure.

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